MULTI-PARAMETER HOMOTOPY FOR FINDING PERIODIC SOLUTIONS OF POWER ELECTRONIC CIRCUITS

*Denise M. Wolf and Seth R. Sanders**

Department of Electrical Engineering and Computer Sciences
University of California, Berkeley CA 94720
e-mail: dwolf@power.berkeley.edu and sanders@eecs.berkeley.edu

ABSTRACT

Commonly used methods for calculating periodic steady state, such as forward integration and shooting, may fail for highly nonlinear circuits with multiple solutions and/or multiple time scales. Homotopy continuation methods, because of their potentially large or global regions of convergence, and suitability for finding multiple solutions, have been applied to the calculation of periodic steady state for such systems.

This paper applies real and complex multi-parameter homotopy to finding periodic solutions of power electronic circuits. We show that multi-parameter homotopy methods can avoid period-doubling and cyclic fold bifurcations along solution paths, and find all stable and unstable periodic solutions along folding or period-doubling paths. We distinguish between *circuit-direct* and *formulation-indirect* homotopy, and show that the latter (with two real parameters) can avoid period-doubling bifurcations, while the former cannot.

1 INTRODUCTION

A circuit is in a periodic steady state if its state $x(t) = x(t+T)$ for all $t > t'$. In this paper we focus on the calculation of periodic steady states, both stable and unstable, of circuits with periodically varying sources and/or switches, a problem of importance in power electronics, control, and communication systems [1].

Commonly used methods for calculating a periodic steady state include forward integration for asymptotically stable solutions, and locally convergent iterative methods such as shooting, finite differences (time domain), and harmonic balance (frequency domain) [1, 2, 3, 4, 5]. However, these techniques may either fail or become impractical for highly nonlinear circuits with characteristics such as multiple solutions and/or multiple time scales, and are not easily adapted to finding multiple solutions. Examples of power electronic circuits with multiple solutions include the feedback controlled buck converter [7] and ferromagnetic circuits [8], to be discussed later in the paper.

Recently, homotopy continuation methods, with their potentially large or global regions of convergence, have been applied to the calculation of periodic solutions of circuits [6]. The idea behind a continuation method is to embed a parameter in the circuit's nonlinear algebraic-differential equations, or in the algebraic formulation associated with a shooting, finite difference, or harmonic balance method. Setting the parameter to zero reduces the problem to a simple one that can be solved easily, or whose periodic solution is known. A periodic solution of the simple problem is the starting point of a continuation path. The set of equations is then continuously deformed into the originally-posed difficult problem. A solution to the difficult problem is obtained by tracing the corresponding continuation path through solution space.

Homotopy continuation methods have been applied to the calculation of periodic steady state and DC operating points of nonlinear circuits via natural maps such as *source and impedance stepping*, and artificial *convex combination homotopy* functions [11, 6]. Varying a circuit parameter may result in qualitative changes in system trajectories of the circuit, called bifurcations. Local dynamic bifurcations of a periodic orbit include cyclic folds and period-doubling bifurcations [10]. Local dynamic bifurcations of periodic orbits can manifest themselves as folding and pitchfork-bifurcating solution paths of algebraic homotopy continuation formulations. Other potential problems include solutions escaping to infinity, abbreviated paths, closely spaced solution curves, and disjoint branches.

In this paper we apply the concept of real and complex multi-parameter homotopy maps and methods, introduced in [14] for finding DC operating points, to finding periodic solutions of power electronic circuits. We explore their potential for avoiding cyclic fold and period-doubling bifurcations along periodic-solution paths, and for finding all solutions emanating from folding or period-doubling paths. Especially of interest is the case where this potential depends on whether the homotopy transformation is applied directly to the circuit, or indirectly to the system of nonlinear algebraic equations derived from a finite difference formulation.

We show that higher dimensional homotopy methods are capable of avoiding bifurcation points and folds along solution paths while tracing all emerging stable and unstable periodic solutions, and thus offer a technique for finding periodic steady states of power electronic circuits that is far more robust than existing methods. We assume simple, isolated periodic solutions in a compact region of state space.

2 MULTI-PARAMETER HOMOTOPY

A simple example of single parameter homotopy, a special case of multi-parameter homotopy, is source stepping combined with finite differences. In this case the homotopy transformation is the scaling of all independent sources by a parameter $\lambda \in \Re$. The original periodically varying

*This work was supported by SRC contract 93-DC-324, and grants from Tandem Computers and the UC Micro Program.

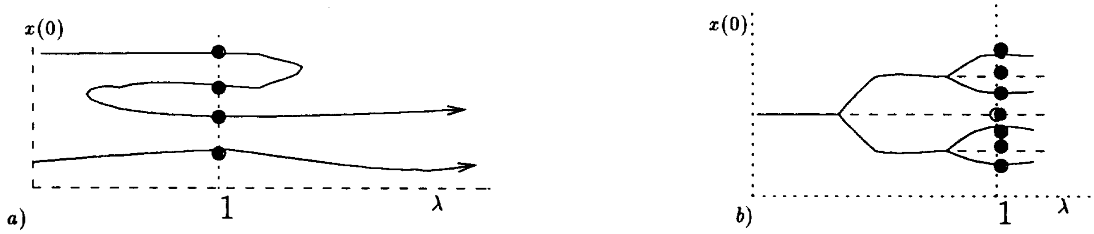


Figure 1: Periodic orbits sampled every T seconds, as a function of λ: a) Cyclic fold bifurcations join multiple solutions. b) Period-doubling cascades join multiple solutions.

state equations[1] $\dot{x} = f(x,t) = f(x,t+T), f : \Re^n \times \Re \rightarrow \Re^n$, describing the circuit are transformed into the system $\dot{z} = h(z,t,\lambda)$, where $h(z,t,1) = f(x,t)$, and the system $\dot{z} = h(z,t,0)$ has a zero solution $z(t) \equiv 0$.

The parameterized two-point boundary value problem $\dot{z}(t,\lambda) = h(z(t,\lambda),t,\lambda), z(0,\lambda) = z(rT,\lambda)$, (a periodic solution with period rT, r$\in Z^+$, is being sought) is posed as a system of algebraic equations via a finite differences formulation [1]. The system $\dot{z} = h(z,t,\lambda)$ can be discretized by the trapezoidal rule, for instance, into

$$z(t_k) \approx z(t_{k-1}) + \Delta t/2[h(z(t_{k-1}),t_{k-1},\lambda) + h(z(t_k),t_k,\lambda)]$$

for k = 1,2..rN (N points every T seconds). Then, to constrain the solution to be periodic, we set $z(t_0) = z(t_{rN})$, and are left with a system of $rN \times n$ nonlinear algebraic equations in $rN \times n + 1$ unknowns written $H(Z_\lambda, \lambda) = 0$. The solution set of the equations $H(Z_\lambda, \lambda) = 0$ consists of curves. A homotopy algorithm traces a curve from $\lambda = 0$ to a circuit solution at $\lambda = 1$, and possibly beyond to search for additional solutions.

Since the solution curves produced by single parameter homotopies like those above can have cyclic fold and period-doubling bifurcation points, which can be problematic for homotopy continuation methods, we study *multi-parameter homotopies* in an effort to avoid these 'bad' points. We are also able to take advantage of the presence of bifurcations along solution paths to find any additional periodic orbits they may serve to connect. Figure 1 illustrates how local bifurcations serve to connect stable and unstable periodic orbits, through cyclic-folding paths and forking period-doubling cascades.

Real m-parameter versions of the homotopy functions described earlier in the section may be obtained by embedding a parameter vector $\bar{\lambda} \in \Re^m$ in the circuit, for instance by scaling different sources independently. The vector-parameterized boundary value problem $\dot{z}(t,\bar{\lambda}) = h(z(t,\bar{\lambda}),t,\bar{\lambda}), z(0,\bar{\lambda}) = z(rT,\bar{\lambda})$, when posed as a system of nonlinear equations $H(Z_{\bar{\lambda}},\bar{\lambda}) = 0$ via a finite difference formulation, has a solution set consisting of locally m-dimensional solution *surfaces*, rather than curves. Similarly, a *complex parameter* version results when λ, the scaling parameter, is made complex. In this case the solution surface will be locally 2-dimensional, consisting of real and imaginary solution components. The idea then is to try and navigate these solution surfaces in such a way that local bifurcation points are avoided, and all locally connected periodic orbits are found. With complex homotopy, paths traced through parameter and solution space contain excursions into complex parameter and solution space, while real homotopies trace paths that remain in real space.

We refer to the above approach to obtaining a homotopy function H, in which the time-invariant homotopy transformation is applied directly to the circuit rather than to the algebraic formulation of the boundary value problem, as *circuit-direct*. Had we first formulated the boundary value problem $\dot{x} = f(x,t), x(0) = x(rT)$, as a system of algebraic equations $F(X) = 0$ via finite differences, and *then* applied the homotopy transformation to the algebraic system $F(X) = 0$ in order to obtain the homotopy function, the approach would be termed *formulation-indirect*.

The next sections discuss the potential of real and complex multi-parameter homotopy methods for avoiding cyclic fold and period-doubling bifurcations of periodic orbits, and finding all emerging stable and unstable periodic orbits along bifurcating paths. The question of whether the fold and bifurcation avoidance results of [14] apply to finding periodic solutions, using circuit-direct and/or formulation-indirect homotopy, is addressed.

3 AVOIDING CYCLIC FOLD BIFURCATIONS

At a cyclic fold bifurcation (CFB), two real periodic orbits, one stable and the other unstable, coalesce and then 'dissapear' as a parameter embedded in the circuit equations is varied monotonically. Power electronic circuits undergoing natural continuation are prone to cyclic fold bifurcations along solution paths. For example, the ferroresonant circuit in Example 1 undergoes two cyclic fold bifurcations as E, the magnitude of the sinusoidal forcing function, is varied from $E = 0$ volts to $E = 140$ volts.

A typical mathematical characterization of a CFB involves the Poincaré map P_λ of a periodic orbit $\tilde{x}_\lambda$ and its eigenvalues, called Floquet multipliers [10]. Conceptually, a Poincaré map is obtained by placing an $n-1$ dimensional hypersurface Σ in $\Re^n$ so that it transversally intersects the periodic orbit $\tilde{x}_\lambda$ exactly once (at p_λ), as shown in Figure 2a. The Poincaré map $P_\lambda : U \rightarrow \Sigma$ sends points in the neighborhood of p_λ on Σ ($q \in U$) to the hypersurface Σ for a first return that matches the system flow. For a periodically forced system, this amounts to sampling the system flow every T seconds.

Since the point $p_\lambda \in \tilde{x}_\lambda$ is a fixed point of the map P_λ ($p_\lambda = P_\lambda(p_\lambda)$), the eigenvalues of the linearization of P_λ at p_λ, $\sigma_i \in \sigma(DP_\lambda(p_\lambda))$, reflect the stability of the fixed point p_λ and its corresponding periodic orbit $\tilde{x}_\lambda$, and determine the occurence of a bifurcation. As long as no eigenvalue is on the unit circle ($|\sigma_i| \neq 1, \forall i$), the periodic orbit $\tilde{x}_\lambda$ is hyperbolic and non-bifurcating. However, when $|\sigma_i| = 1$, the periodic orbit undergoes some type of bifurcation.

[1] A state equation formulation is not necessary. The system of algebraic differential circuit equations derived from modified node analysis or some such method will work in all of the following, with minor modifications.

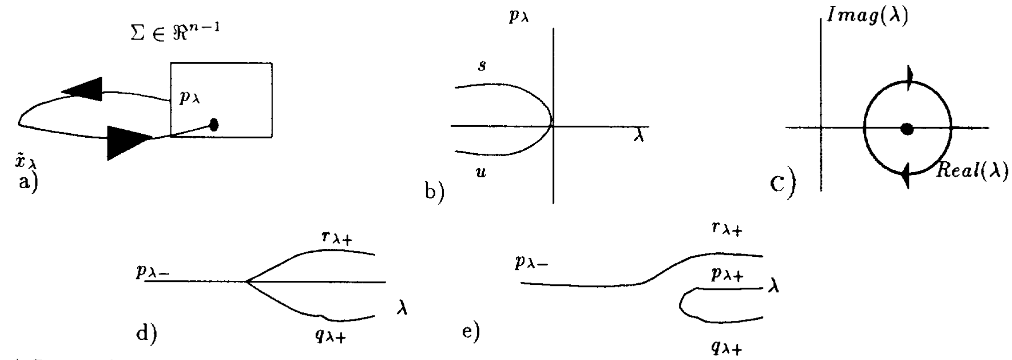


Figure 2: a) Poincaré map. b) CFB fixed point fold. c) Complex CFB avoidance path for Example 2. d) PDB fork in P_λ^2. e) Resoved PDB fork in P_λ^2 (formulation-indirect).

A cyclic fold bifurcation corresponds to a single eigenvalue $\sigma_i \in \sigma(DP_\lambda)$ passing through +1 transversally. At the bifurcating parameter value, the matrices $D_pP_\lambda(p_\lambda) - I$ and $D_\lambda P_\lambda(p)$ drop and maintain rank, respectively, and the solution curve of fixed points p_λ folds, as shown in Figure 2b. We assume that once formulated as a parameterized system of algebraic equations, the periodic-solution finding problem will, for a fine enough discretization, have folding solution paths mirroring any existing CFBs.

Homotopy methods capable of handling solution curves with folds must be able to respond by reversing direction along the parameter axis, a maneuver that can be inefficient for arc length parameterized methods [12] because of the small step sizes required. Switched parameter algorithms [13], while faster than arc length methods, may miss sharp turning points if the step size is not small enough, and can exhibit cyclic behavior near switching points. We are interested in the potential of real and complex multi-parameter homotopy, both circuit-direct and formultion-indirect, for avoiding solution path folds corresponding to cyclic fold bifurcations of periodic orbits.

Since the topic of fold avoidance for multi-parameter homotopy applied to calculating DC operating points was discussed in [14], we address the question of whether there is any qualitative difference, either in real or complex space, between the solution folds one finds in ordinary parameterized systems of algebraic equations, and those that appear in algebraic formulations of two-point boundary value problems as a circuit parameter is varied, reflecting cyclic fold bifurcations. If not, results that apply to fold avoidance in the DC problem [14] will apply to fold avoidance in the periodic-solution finding problem, and there will be no difference between circuit-direct and formulation-indirect homotopy. We demonstrate that the two are locally equivalent.

To show this equivalence, we compare bifurcation normal forms and codimensions. The normal form, or simplest, one dimensional representation of the Poincaré map $x_{k+1} = P_\lambda(x_k)$ in the neighborhood of a CFB, is $x_{k+1} = x_k + x_k^2 + \lambda$ [10], which, for the fixed point $x_{k+1} = x_k = p_\lambda$, is identical to the normal form of the static, generic DC operating point fold discussed in [14], $H(x, \lambda) = x^2 + \lambda = 0$. Both sources of folds are locally codimension one, meaning that a single constraint plus transversality (an eigenvalue passing transversally through +1 for a CFB, and a transversal loss of rank of the Jacobian $D_xH(x, \lambda)$ for the static bifurcation case) locally characterizes their presence. Thus, local results and reasoning that apply to one apply to the other, regardless of whether the homotopy function is circuit-direct or formulation-indirect. We now restate a local version of the results of [14] in the context of periodic orbit tracing and present a circuit example.

Result 1: Real multi-parameter homotopy, applied either to a circuit-direct or a formulation-indirect homotopy function, generally *cannot* avoid cyclic fold bifurcations by locally maneuvering around the parameter value corresponding to the CFB.

Result 2: Complex parameter homotopy, applied either to an analytic circuit-direct or formulation-indirect homotopy function, *can* avoid cyclic fold bifurcations. Generally, folds may be avoided by tracing a closed curve in complex parameter space around the parameter value corresponding to the CFB.

To briefly summarize the reasoning detailed in [14], Results 1 and 2 are based on codimension and normal form arguments. Since folds are codimension one in real parameter space, and codimension two in complex parameter space, adding real parameters to a homotopy function cannot lead to fold avoidance, but complexifying a parameter can lead to fold avoidance. Examining the normal form of a fold, $H(x, \lambda) = x^2 + \lambda = 0$, reveals that tracing a full circle in complex parameter space $\lambda = \epsilon e^{i\theta}, \theta = 0 : 2\pi$, around the fold point $\lambda = 0$ leads to a regular path from $x = \sqrt{\epsilon}$ (on the manifold $x = \sqrt{\lambda}$) to $x = -\sqrt{\epsilon}$ (on the manifold $x = -\sqrt{\lambda}$).

Because CFB's can serve to connect periodic orbits of a parameterized circuit, Result 2 implies that complex homotopy can be used to find all orbits connected by CFB's via regular paths.

Circuit Example 1: The ferroresonant circuit shown in Figure 3a has state equations

$$\begin{aligned}(R_4 + R_5)\dot{q}_1 &= -q_1/C_1 + R_5 g(\phi_2) + e_3(t)\\ (R_4 + R_5)\dot{\phi}_2 &= -R_5 q_1/c_1 - R_4 R_5 g(\phi_2) + R_5 e_3(t)\end{aligned}$$

with nonlinear inductor characteristic $i_2 = g(\phi_2) = a\phi_2 + b\phi_2^3$, a sinusoidal forcing function $e_3(t) = E\cos wt$, and where q_1 is the charge across the capacitor and ϕ_2 is the inductor flux. For the state vector $x = (q_1, \phi_2)$, we refer to the above state equations as $\dot{x} = f(x, t)$. A circuit-direct, real, single parameter homotopy function is obtained by scaling the magnitude of the forcing function

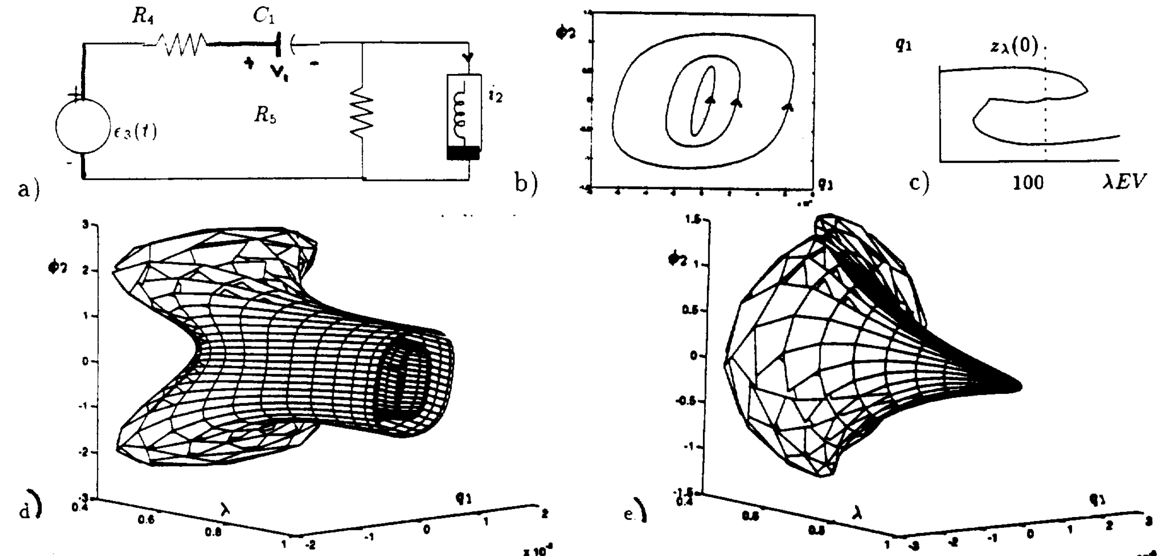


Figure 3: a) Ferroresonant circuit. b) Three periodic orbits at $\lambda E = 100V$. c) Folding solution path. d) Real component of periodic orbit path vs. Real(λ). e) Imaginary component of periodic orbit path vs. Real(λ) .

$e_3(t)$ in $f(x,t)$ by a real parameter λ. The newly parameterized state equations $\dot{z} = h(z,t,\lambda)$ are then formulated, via finite differences with period $2\pi/w$, as a parameterized algebraic system of equations $H(Z_\lambda, \lambda) = 0$, where Z_λ is the N-point discretized T-periodic solution $\tilde{z}_\lambda(\cdot)$ of $\dot{z}(t,\lambda) = h(z(t,\lambda),t,\lambda); z(0,\lambda) = z(rT,\lambda)$. The folding solution path of $H(Z_\lambda,\lambda) = 0$, corresponding to values of λE at which the circuit has first one, $(0 < E_\lambda y)$, then three, and then one T-periodic solution [8], is represented in Figure 3c for the circuit values $R_4 = 50\Omega$, $R_5 = 10\Omega$, $C_1 = 1.69\mu F$, E=100V, (a,b)=(0.03,0.174) and N=40 sample points. The three orbits at E=100V are illustrated in Figure 3b.

A complex parameter homotopy function is obtained from the real homotopy function $H(Z_\lambda,\lambda) = 0$ by making the parameter, and thus the solution vector, complex ($\lambda \in C$). Figures 3d,e show the smooth, fold-free complex solution path (real and imaginary parts of the solution as a function of the real part of λ) from the middle, unstable periodic solution $\tilde{z}_2(\cdot)$ to the outer, stable periodic solution $\tilde{z}_3(\cdot)$ at $\lambda E = 100$ obtained by tracing a full circle in complex parameter space around the CFB point, as shown in Figure 2c. The middle, unstable periodic orbit $\tilde{z}_2(\cdot)$ was obtained in the same manner from $\tilde{z}_1(\cdot)$ by tracing a complex solution curve around the fold point $\lambda E \approx 110V$. Thus all stable and unstable orbits joined by cyclic fold bifurcations on a single branch, $\tilde{z}_1(\cdot)$, $\tilde{z}_2(\cdot)$, and $\tilde{z}_3(\cdot)$, were found via regular paths. ♣

4 AVOIDING PERIOD-DOUBLING BIFURCATIONS

At a period-doubling bifurcation (PDB), the variation of a circuit parameter causes a stable mT-periodic solution to become unstable just as two stable $2mT$-periodic solutions, mT-shifted versions of each other, are created. In the above description, T is the fundamental period of the periodic forcing function in the circuit, and m is a positive integer. Power electronic circuits undergoing natural continuation are prone to period-doubling bifurcations [7, 8]. Figure 4a shows a buck converter circuit from [7] that exhibits period doubling behavior as the amplitude of the input voltage (V_I) is varied from 15.0v to 40.0v. For $V_I < 25.0$v the circuit has a stable periodic orbit with period T, which then bifurcates to two (identical in phase space, but shifted in time) stable periodic orbits of period $2T$ at $V_I \approx 28.0v$. At $V_I \approx 32.0v$ the two $2T$ periodic orbits bifurcate to four stable $4T$ periodic orbits, and so on, until the onset of a chaotic looking waveform at $V_I \approx 40.0v$.

As with the cyclic fold bifurcations discussed in Section 3, Poincaré maps and their eigenvalues are used to characterize period doubling bifurcations. Given a Poincaré map P_λ of a stable periodic orbit $\tilde{x}_\lambda$ of period mT with a fixed point $p_\lambda \in \tilde{x}_\lambda$, the periodic orbit $\tilde{x}_\lambda$ undergoes a period-doubling bifurcation if a single eigenvalue $\sigma_i \in \sigma(D_pP_\lambda(p_\lambda))$ passes transversally through -1 ($\sigma_i = -1, \partial\sigma_i/\partial\lambda \neq 0$). As this eigenvalue passes through the unit circle, the fixed point p_λ reverses its stability, going from stable to unstable as σ_i exits the unit circle. Simultaneously, the second return map P_λ^2 undergoes a fork bifurcation, in which the single fixed point p_λ splits into three fixed points, p_λ, q_λ, and r_λ, with $r_\lambda = P_\lambda(q_\lambda)$, $q_\lambda = P_\lambda(r_\lambda)$ and $p_\lambda = P_\lambda(p_\lambda)$, as shown in Figure 2d. The fixed point p_λ is a point on the now-unstable period-mT orbit $\tilde{x}_\lambda$, while the emerging fixed points r_λ and q_λ are mT-separated points on the emerging stable period-$2mT$ orbit $\tilde{y}_\lambda$.

Analogously, if the periodic solution finding problem is posed as a system of algebraic equations, say via finite differences or shooting *over the fundamental period T*, a smooth well-conditioned solution path approximating $\tilde{x}_\lambda$ exists through a range of parameters including that at which the circuit period doubles. The orbit becomes unstable, but this is not reflected in a static bifurcation of the algebraic system. This is analogous to tracing the fixed point $p_\lambda = P_\lambda(p_\lambda)$ as λ is varied. However, if one traces the period-T solution $\tilde{x}_\lambda$ of a finite differences or shooting formulation over *twice the fundamental period*, $2T$, then a fork bifurcation occurs near the parameter value at which the circuit period-doubles. This is like tracing the fixed point $p_\lambda = P_\lambda^2(p_\lambda)$ as λ is varied. We say 'near' rather than at the PDB point because we assume that they only coincide as h, the discretization increment, goes to zero.

A real, single parameter homotopy algorithm encountering a fork bifurcation point will either fail when the Jacobian of the homotopy function drops rank (unlikely because of sampling and finite precision) or suffer from ill-conditioning in the neighboring region and likely continue along the center branch of the fork corresponding to the unstable period-mT solution. We discuss whether real and/or complex multi-parameter methods, either circuit-direct or formulation-indirect, can avoid the fork bifurcations engendered by period-doubling bifurcations, and, if so, whether all stable and unstable emerging periodic solutions will be accessible.

Since the topic of fork bifurcation avoidance for multi-parameter homotopy applied to calculating DC operating points was discussed in [14], we address the question of whether there is any qualitative difference, either in real or complex space, between the forks one finds in ordinary parameterized systems of algebraic equations, and those that occur in algebraic formulations of two-point boundary value problems as a circuit parameter is varied through a period-doubling bifurcation. As will be shown, the answer is *yes*. Thus, while the discussion and results in [14] on the topic of fork bifurcation avoidance in real and complex space can apply to the periodic solution finding problem when parameters are chosen in a formulation-indirect manner, they do not apply when parameters are chosen circuit-direct.

The idea behind this distinction is that a circuit-direct formulation inherits the bifurcation set characteristics of the boundary-value constrained dynamical system ($p_\lambda = P_\lambda^2(p_\lambda)$ in this case), while a formulation-indirect homotopy function may be chosen to have a bifurcation set that resembles that of the static problem explored in [14]. Since these bifurcation sets can have fundamentally different properties in the neighborhood of a given bifurcation, as they do at a period-doubling engendered fork, bifurcation-avoidance potential and methods are formulation dependent. A summary of results follows.

Result 3: Real 2-parameter homotopy, applied to a formulation-indirect homotopy function ($rT = 2mT$ in Section 2), *can* be used to avoid a PDB point and trace the emerging stable period-$2mT$ orbit. It *cannot* be used to access the continuing unstable period-mT orbit beyond the PDB point without passing through a bifurcation.

Result 4: Real 2-parameter homotopy, applied to a circuit-direct homotopy function ($rT = 2mT$), *cannot* be used (via ϵ-perturbations) to avoid a PDB point and trace either the emerging stable period-$2mT$ orbit or the continuing (unstable) period-mT orbit.

Result 5: Complex parameter homotopy, either circuit-direct or formulation-indirect ($rT = 2mT$), *can* be used to avoid the PDB point and trace the continuing (unstable) period-mT orbit, if the homotopy function is analytic.

We illustrate the difference between the bifurcation avoidance potential of circuit-direct and formulation-indirenct homotopy on the quadratic map $x_{k+1} = P_\lambda(x_k) = \lambda x_k(1 - x_k)$, a simple dynamical system considered representative of generic period-doubling phenomenon [8]. At $\lambda = 3$, the eigenvalue of the linearized map $\sigma = \lambda(1 - 2x_0)$ passes through -1, causing the equilibrium point $x_0 = 1 - 1/\lambda$ to period-double. This period-doubling shows up as a fork bifurcation of an equilibrium point of the second return map P_λ^2, which can be algebraically formulated as the two equations $\lambda x_0(1 - x_0) - x_1 = 0$ and $\lambda x_1(1 - x_1) - x_0 = 0$ after setting $x_0 = x_2$.

We obtain a real, two parameter circuit-direct (map-direct, in this case) homotopy function by first embedding an extra parameter in the quadratic map to get $x_{k+1} = P_{\lambda_1,\lambda_2} = \lambda_1 x_k - \lambda_2 x_k^2$, and then setting $x_0 = x_2$ to get the equations $h_1(x_0, x_1, \lambda_1, \lambda_2) = \lambda_1 x_0 - \lambda_2 x_0^2 - x_1 = 0$ and $h_2(x_0, x_1, \lambda_1, \lambda_2) = \lambda_1 x_1 - \lambda_2 x_1^2 - x_0 = 0$. The equilibrium point $x_0 = x_1 = (\lambda_1 - 1)/\lambda_2$ will still fork-bifurcate as a path is traced through parameter space from $\lambda_1 = \lambda_2 = 3-\epsilon$ to $\lambda_1 = \lambda_2 = 3 + \epsilon$, regardless of how that path is chosen. This is because the eigenvalue of the linearized map, $\sigma = \lambda_1 - 2\lambda_2 x_0$, passes from $\sigma = -1 + \epsilon$ at $\lambda_1 = \lambda_2 = 3 - \epsilon$ to $\sigma = -1 - \epsilon$ at $\lambda_1 = \lambda_2 = 3 + \epsilon$ while remaining on the real line, and thus must pass through -1. Also notice that varying λ_1 and λ_2 at different rates ($\lambda_1 \neq \lambda_2$) does not disrupt the symmetry associated with a fork bifurcation, $h_1(x_0, x_1, \lambda_1, \lambda_2) = h_2(x_1, x_0, \lambda_1, \lambda_2)$.

This reasoning generalizes to an arbitrary number of circuit-direct embedded real parameters in an arbitrary dyamical system exhibiting a generic period-doubling bifurcation, because the scalar constraint $\sigma_i = -1$ forms a locally codimension one set in real parameter space, and thus cannot be locally circumvented (Result 4).

A real, two parameter formulation-indirect homotopy function may be obtained for the quadratic map by first writing the period-two finding problem as a system of algebraic equations, and then embedding independent parameters in each equation to get $h_1(x_0, x_1, \lambda_1) = \lambda_1 x_0(1 - x_0) - x_1 = 0$ and $h_2(x_0, x_1, \lambda_2) = \lambda_2 x_1(1 - x_1) - x_0 = 0$. Notice that at $\lambda_1 = \lambda_2 = 3$ there is a fork bifurcation, but, unlike the circuit-direct case, if $\lambda_1 \neq \lambda_2$ there can be no fork bifurcation. To see this, one can derive the composed map $P_{\lambda_1} \circ P_{\lambda_2}$ and observe that it can be reduced to the form of the codimension-two (locally perturbable) 'ordinary' fork discussed in [14]. Observe that in this case varying λ_1 and λ_2 at different rates ($\lambda_1 \neq \lambda_2$) results in a break of the symmetry associated with a fork bifurcation i.e., $h_1(x_0, x_1, \lambda_1) \neq h_2(x_1, x_0, \lambda_2)$. A path traced through the real parameter plane avoiding the bifurcation point $\lambda_1 = \lambda_2 = 3$ will lead from the equilibrium point $x_0 = x_1$ at $\lambda_1 = \lambda_2 = 2.9$, to the emerging period two orbit, for instance $(x_0, x_1) = (0.7646, 0.5580)$ at $\lambda_1 = \lambda_2 = 3.1$.

Once again, the above reasoning generalizes. A formulation-indirect homotopy function gives rise to ordinary pitchfork bifurcations, which have codimension two bifurcation sets (two constraints locally define the set of parameter values at which the homotopy function fork-bifurcates). Thus, perturbing a formulation-indirect homotopy function around the real parameter vector value at which the fork bifurcation occurs resolves the fork into a smooth regular curve leading from $\tilde{x}_\lambda$, a mT periodic circuit solution, to $\tilde{y}_\lambda$, a $2mT$ periodic circuit solution, and a fold, as shown in Figure 2e and Figure 5a for Example 2.

The circuit interpretation of the real perturbation required to resolve the period-doubling induced fork (as in Figure 2e) is that it must take the form of an ϵ magnitude periodic source added to the circuit with a fundamental period of $2mT$ rather than mT, in order to destroy the symmetry of the problem. Such a perturbed circuit will have no periodic solution with a period less than $2mT$ in the neighborhood, and so cannot period double.

Complex parameter homotopy may be used to avoid the fork bifurcation and continue tracing the mT-period solution. This is a consequence of the codimension two bifurcation set in complex parameter space, and a local separability of the quadratic and linear terms in the fork normal form. In our quadratic example, a path through complex parameter space around $\lambda_1 = \lambda_2 = 3$ may be used to avoid the bifurcation point and access the equilibrium point on the other side of the bifurcation, for either homotopy function. See [14] for a discussion of complex bifurcation avoidance

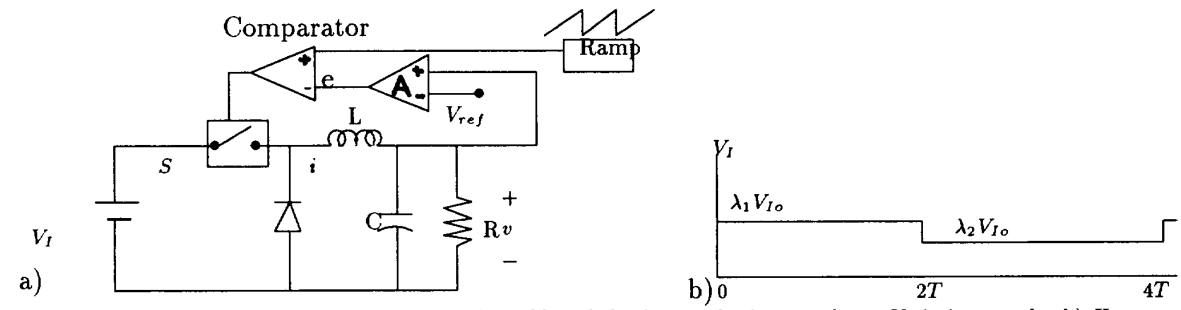


Figure 4: a)The buck converter exhibits period doubling behavior as the input voltage V_I is increased. b) Homotopy parameters embedded in $4T$-periodic input voltage for formulation-indirect, 2-parameter, source stepping homotopy.

and an explanation of why the unstable periodic orbit is not accessible (via a regular path) using real homotopy (Result 3). Example 2 further illustrates this section's results.

Example 2:

As detailed in [7], the state equations describing the operation of the buck converter shown in Figure 4a are

$$\begin{aligned} L\dot{i} &= -v + v_d \\ C\dot{v} &= i - v/R \end{aligned}$$

where $v_d = -Vf$, where Vf is the forward voltage across the diode, when the switch S is open, and $v_d = V_I$, the input voltage, when the switch S is closed. The switch S changes state at times t' satisfying the following equation.

$$v_{con}(t') \stackrel{\text{def}}{=} A(v(t') - V_{ref}) = v_{ramp}(t')$$

As described in [7], the buck converter undergoes a period doubling bifurcation sequence as the input voltage V_I is increased from 20.0v to 40.0v. For instance, at $V_I \approx 28.0$v, a stable periodic orbit with period T bifurcates to two (identical in phase space, but shifted in time) stable periodic orbits of period $2T$ and one unstable orbit of period T, and at $V_I \approx 32.0v$ the two stable $2T$ periodic orbits bifurcate to four stable $4T$ periodic orbits and two unstable $2T$ periodic orbits, and so on.

For switched circuits such as this one, with apriori unknown switching times, a shooting formulation is easier to work with than a finite difference formulation. To derive a two parameter formulation-indirect, shooting based homotopy function $H(z(0,\bar{\lambda}),\bar{\lambda}) = z(0,\bar{\lambda}) - \phi(z(0,\bar{\lambda}),0,4T) = 0$, with state vector is $z = (i,v)$ and parameter vector $\bar{\lambda} = (\lambda_1,\lambda_2)$, we make use of the circuit interpretation of the real perturbation required to resolve the period-doubling induced fork; an ϵ magnitude periodic source added to the circuit with a fundamental period twice that of the orbit being traced. The formulation-indirect homotopy function $H(z(0,\bar{\lambda}),\bar{\lambda}) = 0$ is obtained by embedding two-parameters λ_1 and λ_2 into the now (potentially) periodically varying input voltage, as shown in Figure 4b. When $\lambda_1 \neq \lambda_2$, the input voltage shown in Figure 4b becomes periodic with period $4T$, which is twice the period of the period $2T$ orbit initially being traced in Figure 5a.

Figure 5a shows the regular, bifurcation-free solution path of the formulation-indirect, real, 2-parameter homotopy function $H(z(0,\bar{\lambda}),\bar{\lambda}) = 0$, leading from the stable 2T-periodic solution at $\lambda_1 = \lambda_2 = \lambda_{pdb-}$ to the stable 4T periodic solution at $\lambda_1 = \lambda_2 = \lambda_{pdb+}$ as the the path through real parameter space shown in Figure 5b is traced around the bifurcation value $\lambda_1 = \lambda_2 = \lambda_{pdb}$. In this case the bifurcating parameter value $\lambda_{pdb} \approx 32.0$v. At each step along the path in parameter space shown in Figure 5b, a locally convergent shooting method (over a time period $4T$) is applied to the circuit with input voltage shown in Figure 4. At each new step $(\lambda_1^i,\lambda_2^i)$ along the path, the periodic solution at the parameter value of the previous step, $(\lambda_1^{i-1},\lambda_2^{i-1})$, is used as an initial condition in the shooting method. Notice that the path through the parameter plane avoids the bifurcation value λ_{pdb}. Since the ranks of the Jacobian D_zH and extended Jacobian $[D_zH D_\lambda H]$ of H drop at a period-doubling bifurcation value λ_{pdb}, these ranks can be used to signal the approach of a period-doubling bifurcation.

More specifically, to trace a path in the neighborhood of the bifurcation point $\lambda_1 = \lambda_2 = \lambda_{pdb}$, we let $\lambda_2 = \lambda_1 + \epsilon\sin(\theta)$, $\theta = 0 : \pi$ as λ_1 is varied from λ_{pdb-} to λ_{pdb+}, as shown in Figure 5b. This maneuvering is equivalent to an ϵ-4T-periodic time-varying parameter vector perturbation around λ_{pdb}

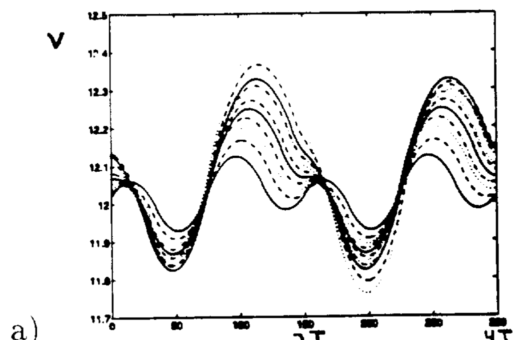
Had we chosen a real circuit-direct homotopy function, such as that obtained by choosing two time-invariant circuit parameters, say V_I and R, as homotopy parameters, the fork bifurcation would not have been avoided, as it was for the formulation-indirect homotopy function in this example. This same process, shooting over twice the period of the orbit being traced, stepping through the parameter plane along the diaganal $\lambda_1 = \lambda_2$, and making half-circle excursions around bifurcation points, can be repeated arbitrarily many times to trace out the stable orbits as they emerge from the period-doubling cascade.

See [15] for an example of the use of complex homotopy in avoiding period-doubling bifurcations and tracing out the unstable cycles of a periodically forced, analytic dynamical system. The unstable orbits may also be traced directly, by using the stable emerging orbits as starting values to paths along the diagonal $\lambda_1 = \lambda_2$, along which shooting is applied over the period of the stable orbit, rather than twice the period of the (originally) stable orbit. This way, the reversal in stability of the orbit at the bifurcating parameter value will not be accompanied by a fork bifurcation along the solution path, and the unstable orbit may be traced to the parameter value of interest. All emerging stable and unstable periodic orbits along a period-doubling cascade can thus be found.

♣

5 SUMMARY

In this paper we focus on the calculation of periodic steady state(s) of power electronic circuits. We contribute multi-



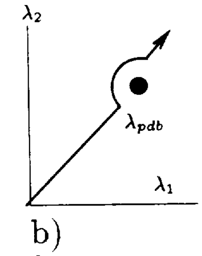


Figure 5: a) Parameter path in (b) leads from a 2T- to a 4T-periodic solution. b) Path through real parameter space around PDB.

parameter homotopy methods and show that they are capable of avoiding bifurcation points and folds along solution paths while tracing all emerging stable and unstable periodic solutions, and thus offer a technique for finding periodic steady states that is far more robust than existing methods.

In particular, we have found that two homotopy parameters (not more), one real and one complex, are enough to ensure the existence of smooth, regular, cyclic fold and period-doubling bifurcation-free periodic-solution paths. In general, no number of added real parameters, in either a circuit-direct or a formulation-indirect homotopy function, can avoid a cyclic fold bifurcation, but a full circle in complex parameter space around the parameter value corresponding to the CFB results in fold avoidance. For a formulation-indirect homotopy function, a half-circle ϵ-excursion in real parameter space around the period doubling bifurcation point will trace emerging stable cycles, but such a strategy will fail for a circuit-direct homotopy function. Complex half-circle ϵ-excursions around period-doubling bifurcation points lead to unstable cycles for either formulation.

REFERENCES

[1] K. Kundert, J. White, A. Sangiovanni-Vincentelli, *Steady-State Methods for Simulating Analog and Microwave Circuits*, 1990.

[2] D.G. Bedrosian, J. Vlach. *An Accelerated Steady-State Method for Networks With Internally Controlled Switches*. IEEE Trans, Circuits Syst., v39, N7:520-530, July, 1992.

[3] R.C. Wong, *Accelerated Convergence to the Steady-State Solution of Closed-Loop Regulated Switching-Mode Systems as Obtained Through Simulation*, IEEE Power Electron. Specialists Conf., pp 682-692, June 1987.

[4] D.C. Hamill, *Improved Time-Domain Simulation of Resonant and Other DC-DC Converters*. IEEE Workshop on Computers in Power Electronics, pp 93-109, 1990.

[5] Y.Kuroe, H. Haneda, and T. Maruhashi, *A General Steady-State Analysis Program ANASP for Thyristor Circuits Based on the Adjoint-Network Approach*, IEEE Power Electron. Specialists Conf., pp 180-189, June 1980.

[6] Y.Kuroe, *Homotopy Applied to Finding Steady-State of Power Electronic Circuits*, Presented at the IEEE Workshop on Computers in Power Electronics, 1990.

[7] J.H. Deane, D.C. Hamill, *Analysis, Simulation and Experimental Study of Chaos in the Buck Converter*, IEEE Power Electron. Specialists Conf., vol 2, pp 491-498, June 1990.

[8] M. Hasler and J. Neirynck, *Nonlinear Circuits*, Artech House, inc, 1986.

[9] E. Allgower, K Georg. *Numerical Continuation Methods: An Introduction*. Springer-Verlag, 1990.

[10] J. Guckenheimer and P. Holmes, *Nonlinear Oscillations,Dynamical Systems, and Bifurcations of Vector Fields*, Springer Verlag, 1983.

[11] Lj. Trajkoric, R. C. Melville, and S.C.Fang. *Finding DC operating points of transistor circuits using homotopy methods*, IEEE Int. Symp on Circuits and Systems, Singapore, June,1991, pp758-761.

[12] L.T. Watson. *Globally convergent homotopy methods: a tutorial*, Appl. Math. and Comp., vol. 31, pp. 369-396, May 1989.

[13] L. O. Chua and A. Ushida. *A switching-parameter algorithm for finding multiple solutions of nonlinear resistive circuits*, Int. j. cir. th eor. appl., 4, 215-239, 1976.

[14] D.M. Wolf and S.R. Sanders. *Multi-Parameter Methods for Finding DC Operating Points of Nonlinear Circuits*, IEEE Int. Symp on Circuits and Systems, Chicago, May, 1993. (SRC ID C93178)

[15] D.M. Wolf and S.R. Sanders. *Multi-Parameter Methods for Finding Periodic Solutions of Nonlinear Circuits*, IEEE Int. Symp on Circuits and Systems, London, June, 1994.