MULTI-PARAMETER HOMOTOPY METHODS FOR FINDING PERIODIC SOLUTIONS OF NONLINEAR CIRCUITS

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ABSTRACT

This paper applies real and complex multi-parameter homotopy to finding periodic solutions of nonlinear circuits. We show, using circuit examples and normal forms coupled with codimension arguments, that multi-parameter homotopy methods can avoid period-doubling and cyclic fold bifurcations along solution paths, and find all solutions along a period-doubling path. We distinguish between *circuit-direct* and *formulation-indirect* multi-parameter homotopy, and show that the latter (with two real parameters) can avoid period-doubling bifurcations, while the former cannot.

1 INTRODUCTION

This paper focuses on the calculation of periodic solutions, both stable and unstable, of circuits with periodically varying sources and/or parameters (e.g. a switch), a problem of importance in power electronics, control, and communication systems [1]. Commonly used methods for calculating periodic steady state include forward integration for asymptotically stable solutions, and locally convergent iterative Newton-based methods such as shooting, finite differences (time domain), and harmonic balance (frequency domain) [1]. However, these techniques may either fail or become impractical for circuits with characteristics such as multiple time scales and/or multiple solutions.

Recently, homotopy continuation methods, with their potentially large or global regions of convergence, have been applied to the calculation of periodic solutions of circuits [8]. The idea behind a continuation method is to embed a parameter in the circuit's nonlinear algebraic-differential equations, or in the algebraic formulation associated with a shooting, finite difference, or harmonic balance method. Setting the parameter to zero reduces the problem to a simple one that can be solved easily, or whose periodic solution is known. The periodic solution of the simple problem is the starting point of a continuation path. The set of equations is then continuously deformed into the originally-posed difficult problem. The solution to the difficult problem is obtained by tracing the corresponding continuation path through solution space.

Varying a circuit parameter may result in qualitative changes in system trajectories of the circuit, called bifurcations. Local dynamic bifurcations of a periodic orbit include cyclic folds and period-doubling bifurcations [11]. Local dynamic bifurcations of periodic orbits can manifest themselves as folding and pitchfork-bifurcating solution paths of algebraic homotopy continuation formulations.

In this paper we apply the concept of real and complex multi-parameter homotopy maps and methods, introduced in [7] for finding DC operating points, to finding periodic solutions of nonlinear circuits. We explore their potential for avoiding cyclic fold and period-doubling bifurcations along periodic-solution paths, and for finding all solutions emanating from a period-doubling path. Especially of interest is the case where this potential depends on whether the homotopy transformation is applied directly to the circuit equations, or indirectly to the system of nonlinear algebraic equations derived from a finite difference or shooting problem formulation. The concepts are illustrated on normal forms, which are the lowest dimensional, simplest, polynomial representations of bifurcating systems [11], and on two circuit examples. We assume simple, isolated periodic solutions in a compact region of state space.

2 MULTI-PARAMETER HOMOTOPY

A simple example of single parameter homotopy, a special case of multi-parameter homotopy, is source stepping combined with finite differences. In this case the homotopy transformation is the scaling of all independent sources by a parameter $\lambda \in [0,1]$. The original periodically varying state equations[1] $\dot{x} = f(x,t) = f(x,t+T), f : \Re^n \times \Re \rightarrow \Re^n$, describing the circuit are transformed into the system $\dot{z} = h(z,t,\lambda)$, where $h(z,t,1) = f(x,t)$, and the system $\dot{z} = h(z,t,0)$ has a zero solution $z(t) \equiv 0$.

The parameterized two-point boundary value problem $\dot{z}(t,\lambda) = h(z(t,\lambda),t,\lambda), z(0,\lambda) = z(rT,\lambda)$, (a periodic solution with period rT, r$\in Z^+$, is being sought) is then posed as a system of algebraic equations via a finite differences formulation [1]. The system $\dot{z} = h(z,t,\lambda)$ can be discretized by the trapezoidal rule, for instance, into

$$z(t_k) \approx z(t_{k-1}) + \Delta t/2[h(z(t_{k-1}),t_{k-1},\lambda) + h(z(t_k),t_k,\lambda)]$$

for k = 1,2..rN (N points every T seconds). Then, to constrain the solution to be periodic, we set $z(t_0) = z(t_{rN})$, and are left with a system of $rN \times n$ nonlinear algebraic equations in $rN \times n + 1$ unknowns written $H(Z_\lambda, \lambda) = 0$. The algebraic equations $H(Z_\lambda, \lambda) = 0$ have a solution set consisting of curves, the characteristics of which will influence any curve tracing algorithm.

A *real m-parameter* version of the above homotopy function may be obtained by embedding a parameter vector $\bar{\lambda} \in \Re^m$ in the circuit, for instance by scaling different sources independently. The vector-parameterized boundary value problem $\dot{z}(t,\bar{\lambda}) = h(z(t,\bar{\lambda}),t,\bar{\lambda}), z(0,\bar{\lambda}) = z(rT,\bar{\lambda})$, when posed as a system of algebraic equations $H(Z_{\bar{\lambda}}, \bar{\lambda}) = 0$, has a solution set consisting of locally m-dimensional solution *surfaces*, rather than curves. Similarly, a *complex parameter* version results when λ, the scaling parameter, is made complex. In this case the solution surface will be locally 2-dimensional, consisting of real and imaginary solution components.

We refer to the above approach to obtaining a homotopy function H, in which the time-invariant homotopy transformation is applied directly to the circuit equations rather than to the algebraic formulation of the boundary value problem, as *circuit-direct*. Had we first formulated the boundary value problem $\dot{x} = f(x,t), x(0) = x(rT)$, as a system of algebraic equations $F(X) = 0$, and *then* applied the homotopy transformation to the algebraic system $F(X) = 0$ in order to obtain the homotopy function, the approach would be termed *formulation-indirect*.

Since both circuit-direct and formulation-indirect multi-parameter homotopy functions have solution surfaces rather than solution curves, there are, if any, infinitely many solution paths

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[1] A state equation formulation is not necessary. The system of algebraic differential circuit equations derived from modified node analysis or some such method will work in all of the following, with minor modifications.

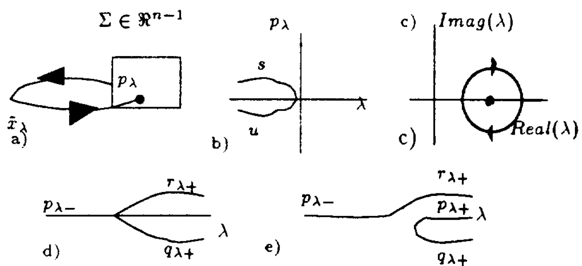

Figure 1: a) Poincaré map. b) CFB fixed point fold. c) Complex CFB avoidance path for Example 2. d) PDB fork in P_λ^2. e) Resoved PDB fork in P_λ^2 (formulation-indirect).

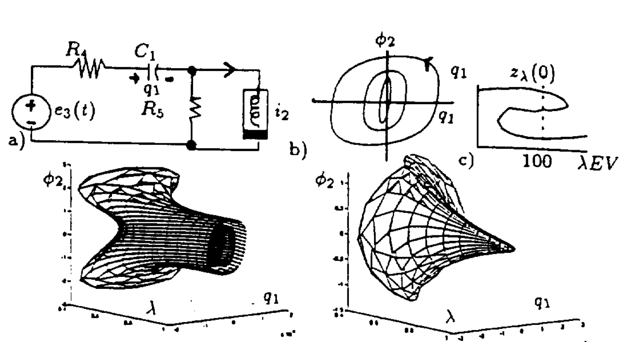

Figure 2: a) Ferroresonant circuit. b) Three periodic orbits at $\lambda E = 100V$. c) Folding solution path. d) Real component of periodic orbit path vs. Real(λ). e) Imaginary component of periodic orbit path vs. Real(λ).

leading from the initial, 'easy' problem to the final, difficult problem of interest, some of which may be better behaved than the solution curves produced by the corresponding single parameter homotopy. The next sections discuss the potential of real and complex multi-parameter homotopy methods for avoiding cyclic fold and period-doubling bifurcations of periodic orbits. The question of whether the fold and bifurcation avoidance results of [7] apply to finding periodic solutions, using circuit-direct and/or formulation-indirect homotopy, is addressed.

3 AVOIDING CYCLIC FOLD BIFURCATIONS

At a cyclic fold bifurcation (CFB), two real periodic orbits, one stable and the other unstable, coalesce and then 'dissapear' as a parameter embedded in the circuit equations is varied monotonically. For example, the ferro-resonant circuit in Example 1 undergoes two cyclic fold bifurcations as E, the magnitude of the sinusoidal forcing function, is varied from $E = 0$ volts to $E = 140$ volts.

A typical mathematical characterization of a CFB involves the Poincaré map P_λ of a periodic orbit $\tilde{x}_\lambda$ and its eigenvalues, called Floquet multipliers [11]. Conceptually, a Poincaré map is obtained by placing an $n-1$ dimensional hypersurface Σ in $\Re^n$ state space so that it transversally intersects the periodic orbit $\tilde{x}_\lambda$ exactly once (at p_λ), as shown in Figure 1a. The Poincaré map $P_\lambda : U \to \Sigma$ sends points in the neighborhood of p_λ on Σ ($q \in U$) to the hypersurface Σ for a first return that matches the system flow. For a periodically forced system, this amounts to sampling the system flow every T seconds.

Since the point $p_\lambda \in \tilde{x}_\lambda$ is a fixed point of the map P_λ ($p_\lambda = P_\lambda(p_\lambda)$), the eigenvalues of the linearization of P_λ at p_λ, $\sigma_i \in \sigma(DP_\lambda(p_\lambda))$, reflect the stability of the fixed point p_λ and its corresponding periodic orbit $\tilde{x}_\lambda$, and determine the occurence of a bifurcation. As long as no eigenvalue is on the unit circle ($|\sigma_i| \neq 1, \forall i$), the periodic orbit $\tilde{x}_\lambda$ is hyperbolic and non-bifurcating. However, when $|\sigma_i| = 1$, the periodic orbit undergoes some type of bifurcation.

A cyclic fold bifurcation corresponds to a single eigenvalue $\sigma_i \in \sigma(DP_\lambda)$ passing through $+1$ transversally. At the bifurcating parameter value, the matrices $D_pP_\lambda(p_\lambda) - I$ and $D_\lambda P_\lambda(p)$ drop and maintain rank, respectively, and the solution curve of fixed points p_λ folds, as shown in Figure 1b. We assume that once formulated as a parameterized system of algebraic equations, the periodic-solution finding problem will, for a fine enough discretization, have folding solution paths mirroring any existing CFBs.

Since the topic of fold avoidance for multi-parameter homotopy applied to calculating DC operating points was discussed in [7], we address the question of whether there is any qualitative difference, either in real or complex space, between the solution folds one finds in ordinary parameterized systems of algebraic equations, and those that appear in algebraic formulations of two-point boundary value problems as a circuit parameter is varied, reflecting cyclic fold bifurcations. If not, results that apply to fold avoidance in the DC problem [7] will apply to fold avoidance in the periodic-solution finding problem, and there will be no difference between circuit-direct and formulation-indirect homotopy. We demonstrate that the two are locally equivalent.

To show this equivalence, we compare bifurcation normal forms and codimensions. The normal form, or simplest, one dimensional representation of the Poincaré map $x_{k+1} = P_\lambda(x_k)$ in the neighborhood of a CFB, is $x_{k+1} = x_k + x_k^2 + \lambda$ [11], which, for the fixed point $x_{k+1} = x_k = p_\lambda$, is identical to the normal form of the static, generic DC operating point fold discussed in [7], $H(x, \lambda) = x^2 + \lambda = 0$. Both sources of folds are locally codimension one, meaning that a single constraint plus transversality (an eigenvalue passing transversally through $+1$ for a CFB, and a transversal loss of rank of the jacobian $D_xH(x,\lambda)$ for the static bifurcation case) locally characterizes their presence. Thus, local results and reasoning that apply to one apply to the other, regardless of whether the homotopy function is circuit-direct or formulation-indirect. We now restate a local version of the results of [7] in the context of periodic orbit tracing and present a circuit example.

Result 1: Real multi-parameter homotopy, applied either to a circuit-direct or a formulation-indirect homotopy function, generally *cannot* avoid cyclic fold bifurcations by locally maneuvering around the parameter value corresponding to the CFB.

Result 2: Complex parameter homotopy, applied either to a circuit-direct or a formulation-indirect homotopy function, *can* avoid cyclic fold bifurcations. Generally, folds may be avoided by tracing a closed curve in complex parameter space around the parameter value corresponding to the CFB.

To briefly summarize the reasoning detailed in [7], Results 1 and 2 are based on codimension and normal form arguments. Since folds are codimension one in real parameter space, and codimension two in complex parameter space, adding real parameters to a homotopy function cannot lead to fold avoidance, but complexifying a parameter can lead to fold avoidance. Examining the normal form of a fold, $H(x, \lambda) = x^2 + \lambda = 0$, reveals that tracing a full circle in complex parameter space $\lambda = \epsilon e^{i\theta}, \theta = 0 : 2\pi$, around the fold point $\lambda = 0$ leads to a regular path from $x = \sqrt{\epsilon}$ (on the manifold $x = \sqrt{\lambda}$) to $x = -\sqrt{\epsilon}$ (on the manifold $x = -\sqrt{\lambda}$).

Circuit Example 1: The ferroresonant circuit shown in Figure 2a has state equations

$$(R_4 + R_5)\dot{q}_1 = -q_1/C_1 + R_5 g(\phi_2) + e_3(t)$$

$$(R_4 + R_5)\dot{\phi}_2 = -R_5 q_1/c_1 - R_4 R_5 g(\phi_2) + R_5 e_3(t)$$

with nonlinear inductor characteristic $i_2 = g(\phi_2) = a\phi_2 + b\phi_2^3$ and a sinusoidal forcing function $e_3(t) = E\cos wt$. For the state vector $x = (q_1, \phi_2)$ we refer to the above state equations as $\dot{x} = f(x,t)$. A circuit-direct, real, single parameter homotopy function is obtained by scaling the magnitude of the forcing function $e_3(t)$ in $f(x,t)$ by a real parameter λ. The newly parameterized state equations $\dot{z} = h(z,t,\lambda)$ are then formulated, via finite differences with period $2\pi/w$ (see Section 2), as a parameterized algebraic system of equations $H(Z_\lambda, \lambda) = 0$, where Z_λ is the N-point discretized T-periodic solution $\tilde{z}_\lambda(\cdot)$ of $\dot{z}(t,\lambda) = h(z(t,\lambda),t,\lambda); z(0,\lambda) = z(rT,\lambda)$. The folding solution path of $H(Z_\lambda, \lambda) = 0$, corresponding to values of λE at which the circuit has first one, then three, and then one T-periodic solution [10], is represented in Figure 2c for the circuit values $R_4 = 50\Omega$,

$R_5 = 10\Omega$, $C_1 = 1.69\mu F$, E=100V, (a,b)=(0.03,0.174) and N=40 sample points. The three orbits at E=100V are illustrated in Figure 2b.

A complex parameter homotopy function is obtained from the real homotopy function $H(Z_\lambda, \lambda) = 0$ by making the parameter, and thus the solution vector, complex ($\lambda \in C$). Figures 2d,e show the smooth, fold-free complex solution path (real and imaginary parts of the solution as a function of the real part of λ) from the middle, unstable periodic solution $\tilde{z}_2(\cdot)$ to the outer, stable periodic solution $\tilde{z}_3(\cdot)$ at $\lambda E = 100$ obtained by tracing a full circle in complex parameter space around the CFB point, as shown in Figure 1c. The middle, unstable periodic orbit $\tilde{z}_2(\cdot)$ was obtained in the same manner from $\tilde{z}_1(\cdot)$ by tracing a complex solution curve around the fold point $\lambda E \approx 110V$. ♣

4 AVOIDING PERIOD-DOUBLING BIFURCATIONS

At a period-doubling bifurcation (PDB), the variation of a circuit parameter causes a stable mT-periodic solution to become unstable just as two stable $2mT$-periodic solutions, mT-shifted versions of each other, are created. In the above description, T is the fundamental period of the periodic forcing function in the circuit, and m is a positive integer.

As with the cyclic fold bifurcations discussed in Section 3, Poincaré maps and their eigenvalues are used to characterize period doubling bifurcations. Given a Poincaré map P_λ of a stable periodic orbit $\tilde{x}_\lambda$ of period mT with a fixed point $p_\lambda \in \tilde{x}_\lambda$, the periodic orbit $\tilde{x}_\lambda$ undergoes a period-doubling bifurcation if a single eigenvalue $\sigma_i \in \sigma(D_p P_\lambda(p_\lambda))$ passes transversally through -1 ($\sigma_i = -1, \partial\sigma_i/\partial\lambda \neq 0$). As this eigenvalue passes through the unit circle, the fixed point p_λ reverses its stability, going from stable to unstable as σ_i exits the unit circle. Simultaneously, the second return map P_λ^2 undergoes a fork bifurcation, in which the single fixed point p_λ splits into three fixed points, p_λ, q_λ, and r_λ, with $r_\lambda = P_\lambda(q_\lambda)$, $q_\lambda = P_\lambda(r_\lambda)$ and $p_\lambda = P_\lambda(p_\lambda)$, as shown in Figure 1d. The fixed point p_λ is a point on the now-unstable period-mT orbit $\tilde{x}_\lambda$, while the emerging fixed points r_λ and q_λ are mT-separated points on the emerging stable period-$2mT$ orbit $\tilde{y}_\lambda$.

Analogously, if the periodic solution finding problem is posed as a system of algebraic equations with $rT = mT$, say via finite differences or shooting (see Section 2), a smooth well-conditioned solution path approximating $\tilde{x}_\lambda$ exists through a range of parameters including that at which the circuit period doubles. The orbit becomes unstable, but this is not reflected in a static bifurcation of the algebraic system. This is analogous to tracing the fixed point $p_\lambda = P_\lambda(p_\lambda)$ as λ is varied. However, if one traces the discretized approximation of the period-mT solution $\tilde{x}_\lambda$ with $r = 2m$ in the algebraic formulation, a fork bifurcation occurs near the parameter value at which the circuit period-doubles. This is like tracing the fixed point $p_\lambda = P_\lambda^2(p_\lambda)$ as λ is varied. We say 'near' rather than at the PDB point because we assume that they only coincide as h, the discretization increment, goes to zero.

A real, single parameter homotopy algorithm encountering a fork bifurcation point will either fail when the jacobian of the homotopy function drops rank (unlikely because of sampling and finite precision) or suffer from ill-conditioning in the neighboring region and likely continue along the center branch of the fork corresponding to the unstable period-mT solution. We discuss whether real and/or complex multi-parameter methods, either circuit-direct or formulation-indirect, can avoid the fork bifurcations engendered by period-doubling bifurcations, and, if so, whether all stable and unstable emerging periodic solutions will be accessible.

Since the topic of fork bifurcation avoidance for multi-parameter homotopy applied to calculating DC operating points was discussed in [7], we address the question of whether there is any qualitative difference, either in real or complex space, between the forks one finds in ordinary parameterized systems of algebraic equations, and those that occur in algebraic formulations of two-point boundary value problems as a circuit parameter is varied through a period-doubling bifurcation. As will be shown, the answer is *yes*. Thus, while the discussion and results in [7] on the topic of fork bifurcation avoidance in real and complex space can apply to the periodic solution finding problem when parameters are chosen in a formulation-indirect manner, they do not apply when parameters are chosen circuit-direct.

The idea behind this distinction is that a circuit-direct formulation inherits the bifurcation set characteristics of the boundary-value constrained dynamical system ($p_\lambda = P_\lambda^2(p_\lambda)$ in this case), while a formulation-indirect homotopy function may be chosen to have a bifurcation set that resembles that of the static problem explored in [7]. Since these bifurcation sets can have fundamentally different properties in the neighborhood of a given bifurcation, as they do at a period-doubling engendered fork, bifurcation-avoidance potential and methods are formulation dependent. A summary of results follows.

Result 3: Real 2-parameter homotopy, applied to a formulation-indirect homotopy function ($rT = 2mT$ in Section 2), *can* be used to avoid a PDB point and trace the emerging stable period-$2mT$ orbit. It *cannot* be used to access the continuing unstable period-mT orbit beyond the PDB point without passing through a bifurcation.

Result 4: Real 2-parameter homotopy, applied to a circuit-direct homotopy function ($rT = 2mT$), *cannot* be used (via ϵ-perturbations) to avoid a PDB point and trace either the emerging stable period-$2mT$ orbit or the continuing (unstable) period-mT orbit.

Result 5: Complex parameter homotopy, either circuit-direct or formulation-indirect ($rT = 2mT$), *can* be used to avoid the PDB point and trace the continuing (unstable) period-mT orbit.

We illustrate the difference between the bifurcation avoidance potential of circuit-direct and formulation-indirenct homotopy on the quadratic map $x_{k+1} = P_\lambda(x_k) = \lambda x_k(1 - x_k)$, a simple dynamical system considered representative of generic period-doubling phenomenon [10]. At $\lambda = 3$, the eigenvalue of the linearized map $\sigma = \lambda(1 - 2x_0)$ passes through -1, causing the equilibrium point $x_0 = 1 - 1/\lambda$ to period-double. This period-doubling shows up as a fork bifurcation of an equilibrium point of the second return map P_λ^2, which can be algebraically formulated as the two equations $\lambda x_0(1-x_0)-x_1 = 0$ and $\lambda x_1(1-x_1)-x_0 = 0$ after setting $x_0 = x_2$.

We obtain a real, two parameter circuit-direct (map-direct, in this case) homotopy function by first embedding an extra parameter in the quadratic map to get $x_{k+1} = P_{\lambda_1,\lambda_2} = \lambda_1 x_k - \lambda_2 x_k^2$, and then setting $x_0 = x_2$ to get the equations $h_1(x_0, x_1, \lambda_1, \lambda_2) = \lambda_1 x_0 - \lambda_2 x_0^2 - x_1 = 0$ and $h_2(x_0, x_1, \lambda_1, \lambda_2) = \lambda_1 x_1 - \lambda_2 x_1^2 - x_0 = 0$. The equilibrium point $x_0 = x_1 = (\lambda_1 - 1)/\lambda_2$ will still fork-bifurcate as a path is traced through parameter space from $\lambda_1 = \lambda_2 = 3 - \epsilon$ to $\lambda_1 = \lambda_2 = 3 + \epsilon$, regardless of how that path is chosen. This is because the eigenvalue of the linearized map, $\sigma = \lambda_1 - 2\lambda_2 x_0$, passes from $\sigma = -1 + \epsilon$ at $\lambda_1 = \lambda_2 = 3 - \epsilon$ to $\sigma = -1 - \epsilon$ at $\lambda_1 = \lambda_2 = 3 + \epsilon$ while remaining on the real line, and thus must pass through -1. Also notice that varying λ_1 and λ_2 at different rates ($\lambda_1 \neq \lambda_2$) does not disrupt the symmetry associated with a fork bifurcation, $h_1(x_0, x_1, \lambda_1, \lambda_2) = h_2(x_1, x_0, \lambda_1, \lambda_2)$.

This reasoning generalizes to an arbitrary number of circuit-direct embedded real parameters in an arbitrary dyamical system exhibiting a generic period-doubling bifurcation, because the scalar constraint $\sigma_i = -1$ forms a locally codimension one set in real parameter space, and thus cannot be locally circumvented (Result 4).

A real, two parameter formulation-indirect homotopy function may be obtained for the quadratic map by first writing the period-two finding problem as a system of algebraic equations, and then embedding independent parameters in each equation to get $h_1(x_0, x_1, \lambda_1) = \lambda_1 x_0(1 - x_0) - x_1 = 0$ and $h_2(x_0, x_1, \lambda_2) = \lambda_2 x_1(1 - x_1) - x_0 = 0$. Notice that at $\lambda_1 = \lambda_2 = 3$ there is a fork bifurcation, but, unlike the circuit-direct case, if $\lambda_1 \neq \lambda_2$ there can be no fork bifurcation. To see this, one can derive the composed map $P_{\lambda_1} \circ P_{\lambda_2}$ and observe that it can be reduced to the form of the codimension-two (locally perturbable) 'ordinary' fork discussed in [7]. Observe that in this case varying λ_1 and λ_2 at different rates ($\lambda_1 \neq \lambda_2$) results in a break of the symmetry associated with a fork bifurcation i.e., $h_1(x_0, x_1, \lambda_1) \neq h_2(x_1, x_0, \lambda_2)$. A path traced through the real parameter plane avoiding the bifurcation point $\lambda_1 = \lambda_2 = 3$ will lead from the equilibrium point

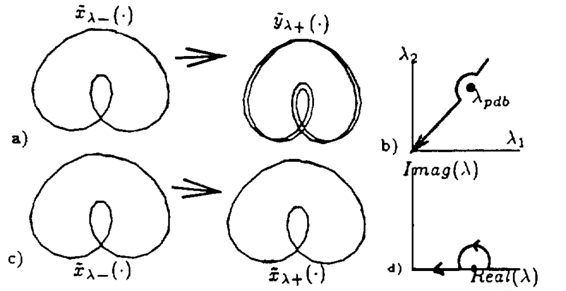

Figure 3: a) Parameter path in (b) leads from a 2T- to a 4T-periodic solution. b) Path through real parameter space around PDB. c) Parameter path (d) leads from a stable 2T- to an unstable 2T-periodic solution. d) Path through complex parameter plane around PDB.

$x_0 = x_1$ at $\lambda_1 = \lambda_2 = 2.9$, to the emerging period two orbit, for instance $(x_0, x_1) = (0.7646, 0.5580)$ at $\lambda_1 = \lambda_2 = 3.1$.

Once again, the above reasoning generalizes. A formulation-indirect homotopy function gives rise to ordinary pitchfork bifurcations, which have codimension two bifurcation sets (two constraints locally define the set of parameter values at which the homotopy function fork-bifurcates). Thus, perturbing a formulation-indirect homotopy function around the real parameter vector value at which the fork bifurcation occurs resolves the fork into a smooth regular curve leading from $\bar{x}_\lambda$, a mT periodic circuit solution, to $\bar{y}_\lambda$, a $2mT$ periodic circuit solution, and a fold, as shown in Figure 1e and Figure 3a for Example 2.

The circuit interpretation of the real perturbation required to resolve the period-doubling induced fork (as in Figure 1e) is that it must take the form of an ϵ magnitude periodic source added to the circuit with a fundamental period of $2mT$ rather than mT, in order to destroy the symmetry of the problem. Such a perturbed circuit will have no periodic solution with a period less than $2mT$ in the neighborhood, and so cannot period double.

Complex parameter homotopy may be used to avoid the fork bifurcation and continue tracing the mT-period solution. This is a consequence of the codimension two bifurcation set in complex parameter space, and a local separability of the quadratic and linear terms in the fork normal form. In our quadradic example, a path through complex parameter space around $\lambda_1 = \lambda_2 = 3$ may be used to avoid the bifurcation point and access the equilibrium point on the other side of the bifurcation, for either homotopy function. See [7] for a discussion of complex bifurcation avoidance and an explanation of why the unstable periodic orbit is not accessible (via a regular path) using real homotopy (Result 3). Example 2 further illustrates this section's results.

Example 2:

The equation $\ddot{y} + k^2 \sin y = \lambda \sin wt$ describes the evolution of the forced oscillations of a pendulum. For the state vector $z = (y, \dot{y})$, we refer to the corresponding parameterized state equations as $\dot{z} = f(z, t, \lambda)$. With $w = 1.3$ and $k = 1$, this system undergoes a period-doubling bifurcation sequence as λ is decreased (see [12] for a full bifurcation diagram). For example, at $\lambda_{pdb} \approx 1.8$, a stable 2T-periodic solution period-doubles to to a stable 4T-periodic solution and an unstable 2T-periodic solution ($T = 2\pi/w$).

Figure 3a represents a regular, bifurcation-free solution path of a formulation-indirect, real, 2-parameter homotopy function $H(Z_{\bar{\lambda}}, \bar{\lambda}) = 0$, leading from the 2T-periodic solution at $\lambda = \lambda_{pdb-}$ to the 4T periodic solution at $\lambda = \lambda_{pdb+}$ as the the path through real parameter space shown in Figure 3b is traced. The homotopy function $H(Z_{\bar{\lambda}}, \bar{\lambda}) = 0$ is obtained by letting $h = f(z(t_{k-1}), t_{k-1}, \lambda_1)$ for $0 \leq t_k \leq 2T$, and $h = f(z(t_{k-1}), t_{k-1}, \lambda_2)$ for $2T \leq t_k \leq 4T$ (see Section 2), which is equivalent to an ϵ-4T-periodic time-varying parameter vector perturbation around λ_{pdb}. For our simple, Newton-corrector path-following simulations we chose $N = 60$ and $\epsilon \leq 0.3$

Had we chosen a time-invariant circuit-direct homotopy function, the fork bifurcation would not have been avoided. With the complex parameter homotopy function obtained by complexifying λ in $H(Z, \lambda)$ ($\lambda_1 = \lambda_2 \in C$), a path through complex parameter space shown in Figure 3d traces a regular complex solution path around the fork bifurcation and leads to the now unstable period-2T orbit at $\lambda = \lambda_{pdb+}$ (Figure 3c). ♣

5 SUMMARY/CONCLUSION

This paper illustrates an approach to reasoning about homotopy transformation choice and path-tracing potential. This approach is based on an analysis of and a comparison between the bifurcations arising in dynamical systems and those that generically occur in relatively unstructured systems of parameterized algebraic equations, such as those found in the DC operating problem. To summarize, we have found that two homotopy parameters (not more), one real and one complex, are enough to ensure the existence of smooth, regular, cyclic fold and period-doubling bifurcation-free periodic-solution paths. In general, no number of added real parameters, in either a circuit-direct or a formulation-indirect homotopy function, can avoid a cyclic fold bifurcation, but a full circle in complex parameter space around the parameter value corresponding to the CFB results in fold avoidance. For a formulation-indirect homotopy function, a half-circle ϵ-excursion in real parameter space around the period doubling bifurcation point will trace emerging stable cycles, but such a strategy will fail for a circuit-direct homotopy function. Complex half-circle ϵ-excursions around period-doubling bifurcation points lead to unstable cycles for either formulation.

REFERENCES

[1] K. Kundert, J. White, A. Sangiovanni-Vincentelli, *Steady-State Methods for Simulating Analog and Microwave Circuits*, 1990.

[2] E. Allgower, K Georg. *Numerical Continuation Methods: An Introduction*. Springer-Verlag, 1990.

[3] Lj. Trajkoric, R. C. Melville, and S.C.Fang. *Finding DC operating points of transistor circuits using homotopy methods*, IEEE Int. Symp on Circuits and Systems, Singapore, June,1991, pp758-761.

[4] R.C. Melville, L. Trajkovic, S-C Fang and L.T. Watson. *Globally Convergent Methods for the DC Operating Point Problem*, TR 90-61, C.S. Dept., Virginia Polytechnic Institute and State University, 1990.

[5] L.T. Watson. *Globally convergent homotopy methods: a tutorial*, Appl. Math. and Comp., vol. 31, pp. 369-396, May 1989.

[6] L. O. Chua and A. Ushida. *A switching-parameter algorithm for finding multiple solutions of nonlinear resistive circuits*, Int. j. cir. th eor. appl., 4, 215-239, 1976.

[7] D.M. Wolf and S.R. Sanders. *Multi-Parameter Methods for Finding DC Operating Points of Nonlinear Circuits*, IEEE Int. Symp on Circuits and Systems, Chicago, May, 1993. Pre-prints of a journal length version submitted to IEEE CAS Trans. available upon request.

[8] Y.Kuroe, *Homotopy Applied to Finding Steady-State of Power Electronic Circuits*, Presented at the IEEE Workshop on Computers in Po wer Electronics, 1990.

[9] J.H. Deane, D.C. Hamill, *Analysis, Simulation and Experimental Study of Chaos in the Buck Converter*, IEEE Power Electron. Specialists Conf., vol 2, pp 491-498, June 1990.

[10] M. Hasler and J. Neirynck, *Nonlinear Circuits*, Artech House, inc, 1986.

[11] J. Guckenheimer and P. Holmes, *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*, Springer Verlag, 1983.

[12] V.I. Gulyayev, A.L. Zubritskaya and V.L. Koshkin, *A Universal Sequence of Period-Doubling Bifurcations of the Forced Oscillations of a Pendulum*, PMM U.S.S.R, Vol. 53, No. 5, pp.561-565, 1989.