Justification of the Describing Function Method for Periodically Switched Circuits

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Abstract

This paper studies the describing function method for analysis of periodically switched circuits, such as those used in power electronic applications. Typical power circuit models have nonlinear elements that do not satisfy a Lipschitz continuity condition. As a result, classical averaging theory is not generally applicable to these switched systems. Furthermore, as a result of the nonsmooth characteristics of circuit nonlinearities, previously developed justifications for the describing function method are also not applicable. The present paper develops a justification for the describing function method that relies on the incrementally passive characteristics of the network elements comprising typical power electronic circuits.

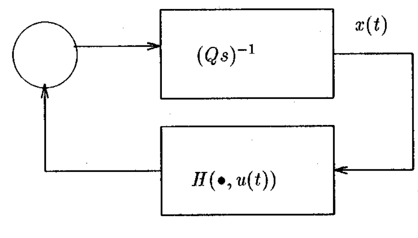

Figure 1: Block Diagram for Partitioned Circuit

1 Introduction

This paper studies the describing function method for analysis of periodically switched circuits, such as those used in power electronic applications. Typical power circuit models have nonlinear elements that do not satisfy a Lipschitz continuity condition. As a result, classical averaging theory [13, 14, 15] is not generally applicable to these switched systems. Further, the elegant justification for the describing function method given by Bergen and Franks [17] is not applicable. The results developed in the above mentioned references are based on Lipschitz conditions on nonlinear elements or system nonlinearities. In contrast, the present paper develops a justification for the describing function method that relies on the incrementally passive characteristics of the network elements comprising typical power electronic circuits.

There is a large and growing body of literature on averaging techniques for power electronic circuits. See for example [1, 2, 3, 4, 20] and also [5, 6, 7, 8] for results on synthesis of averaged circuits that are realizations of the averaged state-space equations. The results reported here offer a partial justification for the generalized averaging method reported in [4]. In particular, the generalized averaging method extends the describing function method for non-steady-state analysis. Here, attention is focused on the steady state.

Nonunique limit cycle and/or chaotic behavior may be undesirable in power electronic circuits, and these phenomena have attracted some attention recently [9, 10, 11]. Reference [19] addressed the issues of existence, uniqueness, and stability of limit cycles in power circuits. The present paper addresses computational methods for approximating limit cycles.

2 The Describing Function Method

Consider the partition of a power electronic circuit into the interconnection of a reactive n-port and a resistive n-port. Suppose all reactive elements are linear and passive. In this case, the circuit can be modeled with the block diagram shown in Figure 1. The block labeled $(Qs)^{-1}$ is a hybrid model for the linear reactive n-port, while the block labeled $H(\bullet, u(t))$ is a hybrid model for the switched resistive n-port. The matrix Q, which need not be diagonal, contains the inductance and capacitance values. The describing function method is based on a Fourier series representation for the state vector of the form

$$x(t) = \sum_k \langle x\rangle_k e^{jk\omega t} \tag{1}$$

where $\omega = 2\pi/T$ corresponds to the frequency of excitation. The Fourier coefficients are determined by

$$\langle x\rangle_k = \frac{1}{T}\int_0^T x(t)e^{-jk\omega t}\,dt\ . \tag{2}$$

In the Fourier domain, a limit cycle is determined by the infinite set of equations that describe harmonic balance:

$$0 = -jkwQ\langle x\rangle_k + \langle H(x,u)\rangle_k \tag{3}$$

for all integers k. The quantity $\langle H(x,u)\rangle_k$ can sometimes be evaluated by consulting a table of describing functions [16]. The essence of the describing function method lies in approximating a limit cycle by truncating the infinite set of equations (3) so that only the relatively large harmonic coefficients are maintained in the calculation. The next section develops a justification for this method in the context of power electronic circuits. In particular, explicit error bounds are computed.

3 Justification

Suppose that in carrying out the describing function analysis, one includes terms corresponding to a certain finite subset S of the infinite set of harmonic coefficients. One would then obtain the approximation

$$\tilde{x}(t) = \sum_{k\in S}\langle\tilde{x}\rangle_k e^{jk\omega_s t} \tag{4}$$

for $x(t)$. The question addressed here is how well $\tilde{x}(t)$ approximates $x(t)$, or more appropriately $Px(t)$, where the operator P is defined by

$$Px(t) = \sum_{k\in S}\langle x\rangle_k e^{jk\omega t}. \tag{5}$$

Note that P projects a periodic waveform onto the harmonic components determined by $k \in S$. The remainder of this development is aimed at determining bounds on $\tilde{x}(t) - Px(t)$ and on $\overline{P}x(t)$, where $\overline{P} = I - P$. The approach for this follows the method of Bergen and Franks [17] and Mees [22, 21] where a system is split into two subsystems - one that governs the behavior of the fundamental coefficients used to construct $Px(t)$, and one that governs the behavior of the parasitic coefficients which result in the waveform $\overline{P}x(t)$.

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Consider a circuit built from ideal sources, incrementally passive resistive elements, ideal switches, and linear passive reactive elements. Partition the circuit as shown in Figure 1 where the reactive elements are extracted from the circuit. Associated with each reactive element is some intrinsic damping. This damping, which is typically a result of the physical construction of the element, guarantees that the linear reactive block has finite gain. No assumption of finite incremental gain is needed for the nonlinear block. Given the partition described here, the system is modeled as the feedback interconnection of a block with transfer function $G(s) = [Q(s+\alpha)]^{-1}$ and a memoryless nonlinearity described by $H(\bullet, u(t))$ which corresponds to the hybrid description of the multiport network obtained after extracting the damped reactive elements. This hybrid model is guaranteed to exist if the circuit has a well defined state-space model. Note that the linear reactive subnetwork has finite gain and is incrementally passive, while the nonlinear subnetwork is incrementally passive since it results from the interconnection of incrementally passive circuit elements. See [12] for a discussion of passivity properties of nonlinear circuits.

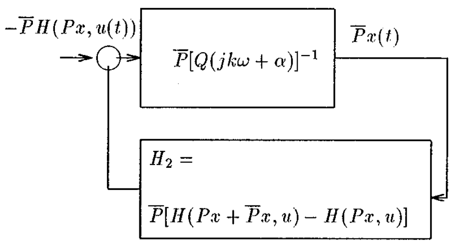

Figure 2: Block Diagram for Behavior of Parasitic Components with Fundamental Components as Parameters

The system waveforms can be split into fundamental and parasitic components. The modified block diagram of Figure 2 models the behavior of the parasitic components of the system waveforms $\overline{P}x(t)$ with the viewpoint that the fundamental components are parameters. The purpose of this description is to study the dependence of the parasitic components on the fundamental components. It can easily be seen that the element $\overline{P}[Q(jk\omega + \alpha)]^{-1}$ in the diagram is incrementally passive and has finite gain since the corresponding element in Figure 1 has these properties. Furthermore, the gain of this element is reduced as more coefficients are included in the fundamental set because of the low pass nature of $[Q(s + \alpha)]^{-1}$. Suppose the gain (induced norm) of this element is γ. It is also true that the element H_2 is incrementally passive as a result of the property of the analogous element of Figure 1. Suppose that H_2 satisfies the condition

$$\{H_2x - H_2x', x - x'\} \geq \epsilon \|H_2x - H_2x'\|^2 \qquad (6)$$

with real $\epsilon > 0$. Here, $\{x, y\} = \frac{1}{T}\int_0^T x^*(t)y(t)\,dt$ with * indicating conjugate transpose and $\|x\| = \sqrt{\{x, x\}}$. Note that the norms of all operators considered in this paper are defined by the induced norm corresponding to $\|\bullet\|$. The condition (6) is equivalent to a strict incremental passivity condition on H_2^{-1}, if this inverse exists. Given these conditions, we can apply a version of the incremental passivity theorem [18]. This theorem is reproduced in the appendix. We obtain

$$\|\overline{P}x\| \leq \gamma(2 + \frac{\gamma}{\epsilon})\|\overline{P}H(Px, u(t))\|. \qquad (7)$$

This gives an explicit bound on the magnitude of the parasitic terms in $x(t)$ in terms of the harmonics generated by the nonlinearity $H(\bullet, u(t))$ operating on the fundamental components. Note that the bound is obtained through passivity properties, whereas [17] developed a similar bound through a small gain argument. In our problem, the small gain theorem cannot be applied since nonlinear elements that are not Lipschitz continuous may be present.

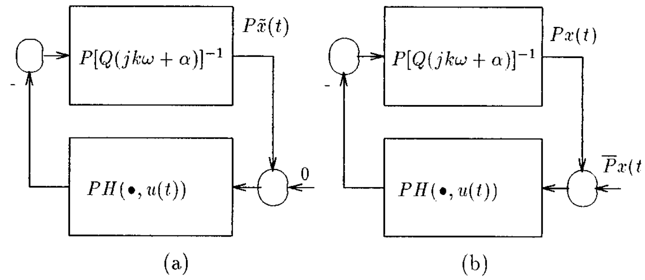

Figure 3: (a) Block Diagram for "Averaged Model" that Ignores Parasitic Terms, and (b) Block Diagram for Fundamental Components with Parasitic Terms as Perturbation

Next consider the block diagrams of Figure 3. Another application of the passivity theorem allows us to generate a bound on the error in the fundamental terms incurred by ignoring the parasitic terms. We obtain

$$\|Px - P\tilde{x}\| \leq \frac{\gamma_0}{\epsilon}\|\overline{P}x\| \qquad (8)$$

where γ_0 is the gain of the operator $P[Q(jk\omega + \alpha)]^{-1}$. Note that if $\overline{P}x$ comprises the exact parasitic terms, then $Px - P\tilde{x}$ is precisely the error incurred in the fundamental terms by using the averaged model.

By combining (7) and (8), we obtain

$$\|Px - P\tilde{x}\| \leq \frac{\gamma_0\gamma}{\epsilon}(2 + \frac{\gamma}{\epsilon})\|\overline{P}H(Px, u(t))\|. \qquad (9)$$

This is a bound on the error in the fundamental components incurred by using the averaged model, in terms of the harmonics generated by the actual fundamental terms. One can argue from this calculation that the approximation error in (9) decreases as more harmonic components are included in the describing function analysis, that is, as more harmonic components are viewed as fundamental components. The reasons for this are: (i) the complementary projection operator $\overline{P}$ projects onto smaller subspaces as the number of fundamental harmonics increases, and (ii) the gain γ of the linear block applied to the parasitic harmonics decreases as the number of fundamental harmonics increases.

The bound (9) can be presented in a more convenient form provided the absolute gain of $H(\bullet, u(t))$ is bounded, i.e. provided

$$\|H(x, u(t))\| \leq k_1\|x\| + k_2 \qquad (10)$$

for some nonnegative real constants k_1 and k_2. This condition does not require that $H(\bullet, u(t))$ have finite incremental gain. Then, one obtains

$$\|\overline{P}H(Px, u(t))\| \leq k_1\|Px - P\tilde{x}\| + k_1\|P\tilde{x}\| + k_2 \,. \qquad (11)$$

Combining this relationship with (9), one finds

$$\|Px - P\tilde{x}\| \leq \frac{\gamma_0\gamma}{\epsilon}(2 + \frac{\gamma}{\epsilon})(k_1\|Px - P\tilde{x}\| + k_1\|P\tilde{x}\| + k_2) \,. \qquad (12)$$

Then, if the quantity $\delta = \frac{\gamma_0\gamma}{\epsilon}(2 + \frac{\gamma}{\epsilon})k_1$ is less than one, one finally obtains

$$\|Px - P\tilde{x}\| \leq \frac{\delta}{1 - \delta}(\|P\tilde{x}\| + \frac{k_2}{k_1}) \,. \qquad (13)$$

Note that the quantity δ can always be made less than one by including a sufficient number of harmonic components as fundamental components because of the low pass nature of the linear reactive subnetwork. By combining (7), (11), and (13), one can write

$$\|\overline{P}x\| \leq \gamma(2 + \frac{\gamma}{\epsilon})\frac{1}{1 - \delta}(k_1\|P\tilde{x}\| + k_2) \,. \qquad (14)$$

The results (13) and (14) are quite convenient since these results give bounds on the error incurred with the describing function method in terms of quantities computed with the describing function method. See the example below for details on the application.

4 Example

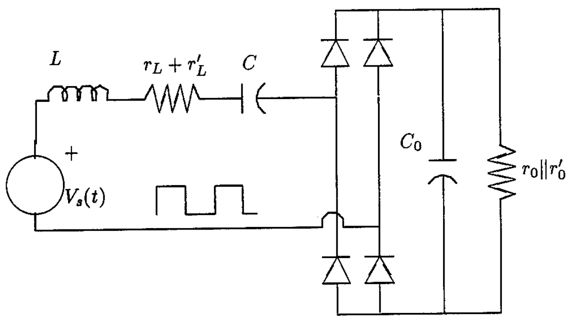

Figure 4: Series Resonant Converter

To illustrate the application of the theory developed in the previous section, we consider the example of the series resonant converter of Figure 4. The first step is to partition the circuit into a linear reactive 2-port and a nonlinear, time-varying resistive 2-port as shown in Figure 5. Note that we have divided the series resistance $r_L + r_L'$ into a damping term associated with the resonant tank and a term associated with the nonlinear resistive network. A similar step has been taken with the output load resistor. We have also lumped the resonant $L - R - C$ network into a single port, rather than complicating the analysis by including a port for each of the reactive elements. With the indicated partition, the linear reactive network is described by

$$\begin{bmatrix} -i_1\sqrt{R_1} \\ v_2/\sqrt{R_2} \end{bmatrix} = \begin{bmatrix} \frac{sCR_1}{s^2LC+sr_LC+1} & 0 \\ 0 & \frac{r_0/R_2}{sC_0r_0+1} \end{bmatrix} \begin{bmatrix} v_1/\sqrt{R_1} \\ -i_2\sqrt{R_2} \end{bmatrix} \quad (15)$$

where $\sqrt{R_1}$ and $\sqrt{R_2}$ are arbitrary scaling factors. The scaling factors are needed to give meaning to the various constants that are to be calculated. We obtain immediately that

$$\gamma_0 \leq max(R_1/r_L, r_0/R_2) . \quad (16)$$

The nonlinear resistive network is described by

$$\begin{bmatrix} v_1/\sqrt{R_1} \\ i_2\sqrt{R_2} \end{bmatrix} = \begin{bmatrix} r_L'/R_1 & \sqrt{R_2/R_1}sgn(i_1) \\ -\sqrt{R_2/R_1}sgn(i_1) & R_2/r_0' \end{bmatrix} \begin{bmatrix} i_1\sqrt{R_1} \\ v_2/\sqrt{R_2} \end{bmatrix} + \begin{bmatrix} -V_s(t)/\sqrt{R_1} \\ 0 \end{bmatrix} \quad (17)$$

where $sgn(\bullet)$ is the sign function. It turns out that for this particular nonlinear model, it is possible to compute the inverse and to obtain the constant ϵ. In particular, we compute

$$\epsilon = \frac{min(R_2/r_0', r_L'/R_1)}{(R_2/R_1)(1 + r_L'/r_0')} . \quad (18)$$

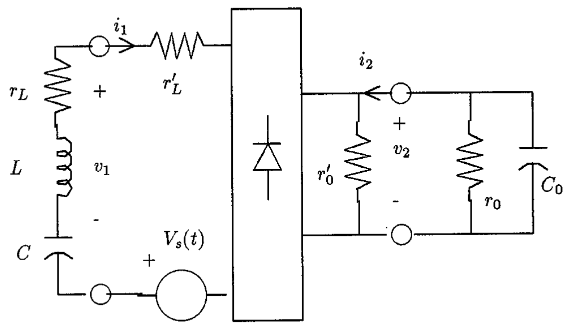

Figure 5: Partition of Series Resonant Converter

Note that it is possible to pick R_1, R_2, and the ratios r_L/r_L' and r_0'/r_0 to make the bound as tight as possible. For the purposes here, we shall make rather arbitrary choices for these parameters to simplify the calculations. We take

$$R_2 = r_0' \quad (19)$$
$$R_1 = r_L' \quad (20)$$
$$r_0' = r_0 \quad (21)$$
$$r_L' = r_L \quad (22)$$

We immediately obtain $\gamma_0 \leq 1$. In a realistic design that attains a reasonable level of power efficiency, it must be true that $r_0 >> r_L$. Let us assume $r_0 = 9r_L$ which then yields $\epsilon = 0.1$ and $k_1 = \sqrt{10}$. The parameter k_2 is then determined to be $k_2 = \|V_s(t)/\sqrt{r_L}\|$. The gain parameter γ depends upon how many harmonic coefficients are included as fundamental components in the describing function analysis. If we include the DC and first harmonic terms, we obtain the estimate $\gamma \approx 0.01$. The rational for this estimate is that at the frequency of the second harmonic, the admittance of the $L - R - C$ network is controlled approximately by a one-pole roll-off and the impedance of the $R - C$ network is also controlled by a one-pole roll-off. For the $L - R - C$ network, we estimate the admittance as $\frac{1}{2}\frac{r_L}{\sqrt{L/C}}$. If the quality factor Q of this resonant network is on the order of 50 (which is reasonable), we find the scaled admittance of this element to be approximately 0.01. For the $R - C$ network, the scaled impedance will have a magnitude of approximately 0.01 at the second harmonic frequency if the r_0C_0 time constant is 50 times $\sqrt{LC}$. This corresponds to a reasonable design. The resulting value of δ is 0.66 which is less than one. The bounds obtained take the form:

$$\|Px - P\tilde{x}\| \leq 2.0(\|P\tilde{x}\| + \frac{1}{\sqrt{10}}\|V_s(t)/\sqrt{r_L}\|) \quad (23)$$

and

$$\|\bar{P}x\| \leq 0.18\|P\tilde{x}\| + 0.06\|V_s(t)/\sqrt{r_L}\| . \quad (24)$$

These are bounds on the error incurred with the describing function method that are computed directly in terms of quantities evaluated when carrying out the describing function analysis. The bounds become tighter if additional harmonics are included in the analysis.

5 Conclusion

This paper offers a justification for the use of the desribing function method for periodically switched circuits. In contrast to previous work, nonlinear elements that do not satisfy a Lipschitz condition can be included in the analysis. Furthermore, the analysis presented here generates explicit bounds on the error incurred with the desribing function method.

A Incremental Passivity Theorem

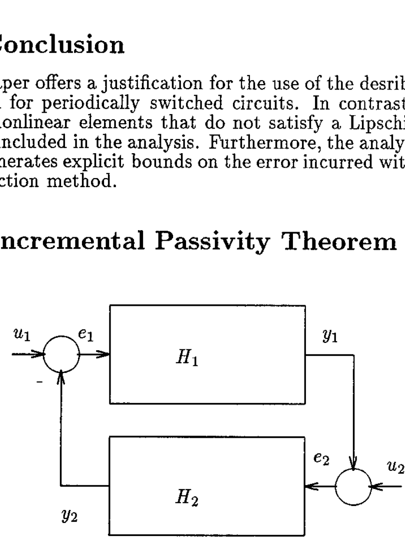

Figure 6: Feedback System

Theorem A.1 (Incremental Passivity Theorem) *Suppose that the operator labeled H_1 in Figure 6 has finite incremental gain γ and is incrementally passive. Suppose that H_2 is incrementally passive, and satisfies $\{H_2x - H_2x', H_2x - H_2x'\} \geq \epsilon\|H_2x - H_2x'\|^2$ for $\epsilon > 0$, i.e. H_2^{-1} is strictly incrementally passive if it exists. Then, (i) for any inputs u_1 and u_2 with bounded norm, the outputs e_1, e_2, y_1, y_2 are unique and have bounded norms, and (ii) the mapping $(u_1, u_2) \to (e_1, e_2, y_1, y_2)$ has bounded gain.*

For a detailed proof of a version of this theorem, see [18]. The important part of the proof that establishes uniqueness also generates the bounds used in Section 3. In particular, by considering two sets of inputs (u_1, u_2) and (u_1', u_2'), we obtain

$$\epsilon\|e_1 - e_1'\|^2 \leq (2\epsilon + \gamma)\|u_1 - u_1'\| \cdot \|e_1 - e_1'\| +$$
$$\|u_2 - u_2'\| \cdot \|e_1 - e_1'\| + \|u_2 - u_2'\| \cdot \|u_1 - u_1'\| \quad (25)$$

By considering the case where $u_1 = u_1'$, we obtain the bounds

$$\|e_1 - e_1'\| \leq 1/\epsilon\|u_2 - u_2'\| \quad (26)$$
$$\|e_2 - e_2'\| \leq \frac{\gamma}{\epsilon}\|u_2 - u_2'\| . \quad (27)$$

By considering the case where $u_2 = u_2'$, we obtain the bounds

$$\|e_1 - e_1'\| \leq (2 + \frac{\gamma}{\epsilon})\|u_1 - u_1'\| \quad (28)$$
$$\|e_2 - e_2'\| \leq \gamma(2 + \frac{\gamma}{\epsilon})\|u_1 - u_1'\| . \quad (29)$$

These are the relationships used in Section 3.

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