Generalized In-Place Circuit Averaging *

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Abstract

The method of in-place circuit averaging has been successfully applied to pulse-width modulated (PWM) power electronic circuits, beginning with the work of Wester and Middlebrook [1]. Recent work by Rim and Cho [2] and others have extended the approach for resonant type circuits. This paper presents a unified approach to in-place circuit averaging that is applicable to resonant type circuits as well as PWM circuits. Furthermore, the present approach allows the refinement of an averaged circuit model to obtain an arbitrary degree of accuracy. The method is illustrated with SPICE runs on a number of examples including both PWM and resonant type circuits.

1 Introduction

The method of in-place circuit averaging has been successfully applied to pulse-width modulated (PWM) power electronic circuits, beginning with the work of Wester and Middlebrook [1]. For a periodically switched circuit with period T, the method firstly involves the application of the one-cycle averaging operator

$$\langle x\rangle(t) = \frac{1}{T}\int_{t-T}^{t} x(s)\,ds \tag{1}$$

to each circuit branch variable $x(t)$. Secondly, appropriate "averaged" circuit elements that are consistent with the averaged branch variables are synthesized. The procedure is illustrated in Figure 1. The strengths of the method are in permitting direct analysis on a given circuit rather than on a state space equation, and in the possibility of using circuit simulation packages such as SPICE. Numerous authors [3, 4, 5, 6, 7] have refined the method in various ways, and recently, some approaches for constructing averaged circuit models for resonant type converter circuits have emerged [2, 8, 9]. This paper presents a unified approach to in-place circuit averaging that is applicable to resonant type circuits as well as PWM circuits. Furthermore, the present approach allows the refinement of any given model to obtain a greater degree of accuracy. With this technique, averaged circuit models can be entered directly into circuit simulators such as SPICE. In most cases, the techniques outlined in this paper permit automated synthesis of the averaged circuit from the original SPICE deck. Often the simulation times for the averaged model are far shorter than those for the underlying switched circuit.

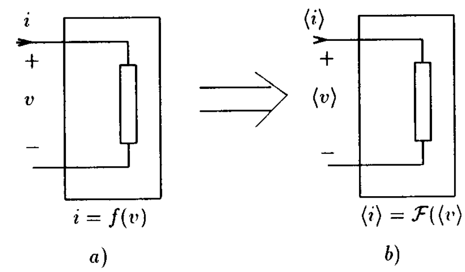

Figure 1: (a) Actual Circuit Element and (b) Averaged Circuit Element

The averaging technique considered in this paper is based on the Fourier series averaging method developed in [10]. This averaging method is based on the fact that a circuit waveform $x(\bullet)$ can be approximated over the interval $(t-T, t]$ to arbitrary accuracy with a Fourier series representation of the form

$$x(t-T+s) = \sum_k \langle x\rangle_k(t) e^{jk\omega_s(t-T+s)} \tag{2}$$

where the sum is over integers k, $\omega_s = 2\pi/T$, $s \in (0, T]$, and the $\langle x\rangle_k(t)$ are complex Fourier coefficients. These Fourier coefficients are functions of time and are determined by

$$\langle x\rangle_k(t) = \frac{1}{T}\int_0^T x(t-T+s)e^{-jk\omega_s(t-T+s)}\,ds. \tag{3}$$

for the k-th coefficient which will also be referred to as the index-k average. To reconstruct $x(t)$ from the coefficients $\langle x\rangle_k(t)$ we apply (2) with $s = T$, yielding

$$x(t) = \langle x\rangle_0 + 2\sum_{k=1}^{k=\infty}\left[Re\{\langle x\rangle_k\}cos(k\omega_s t) - Im\{\langle x\rangle_k\}sin(k\omega_s t)\right] \tag{4}$$

where the t-dependence in the $\langle x\rangle_k$ is omitted for clarity.

In the context of a particular circuit, an approximate averaged representation for each branch variable consists of a

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subset of the Fourier coefficients (index-k averages). The selection of this subset is determined by the dominant harmonic content of the circuit waveforms. For instance, in a series resonant DC-DC converter, the index-1 averages would be selected for the resonant tank variables, whereas the index-0 averages would be selected for the load side elements. In a PWM converter, one would normally use the index-0 averages to obtain a low frequency approximate model. These models can then be refined by including additional coefficients. This procedure is illustrated in Section 3 for a PWM up-down converter and for a DC-DC series resonant converter.

2 Averaged Circuit Elements

The following is an outline of the application of the averaging operation to various circuit elements.

Linear Elements

The branch variables for a linear resistive element are governed by ohm's law $v = Ri$. In applying the index-k average, we make use of linearity to obtain

$$\langle v\rangle_k = R\,\langle i\rangle_k. \tag{5}$$

The voltage and current through an inductor are related by $v = L\frac{d}{dt}i$. By differentiating (3) with respect to time, the averaged relation for an inductor is determined to be:

$$L\frac{d}{dt}\langle i\rangle_k = -j\,k\,\omega_s\,L\langle i\rangle_k + \langle v\rangle_k. \tag{6}$$

By inspection of the above equation, it is easy to see that the index-k averaged model for the inductor is none other than the original inductor in series with a resistor of *complex* value $jk\omega_s L$. Multiport inductors can be treated in a similar manner.

Similarly, the index-k average for a capacitor is:

$$C\frac{d}{dt}\langle v\rangle_k = -j\,k\,\omega_s\,C\langle v\rangle_k + \langle i\rangle_k. \tag{7}$$

In this case it is easy to verify that the index-k averaged model of a capacitor consists of an admittance of complex value $jk\omega_s C$ in parallel with the original element.

Figure 2 summarizes the relationship between the averaged and non-averaged linear circuit elements. Note that real valued variables are being replaced by complex valued coefficients (for $k \neq 0$). This is precisely the replacement suggested in [2].

Nonlinear and Time-Varying Elements

Resistor Given a current controlled nonlinear resistor characterized by $v = f(i)$, application of the index-k average yields $\langle v\rangle_k = \langle f(i)\rangle_k$. Several techniques are available for approximating $\langle f(i)\rangle_k$. In some cases, describing function techniques [10, 11] are most useful. The basic premise of the

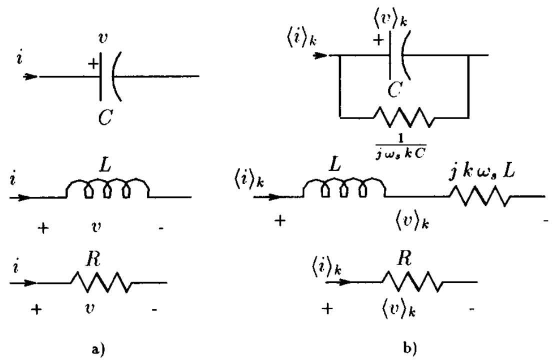

Figure 2: Relationship between (a) Underlying Elements and (b) Index-k Averaged Elements

sinusoidal describing function method is that the current $i(t)$ is well approximated as a sinusoidal waveform of frequency $k\omega_s$. Then, $\langle f(i)\rangle_k$ is approximated by the k-th harmonic component of $f(i)$. The describing function method can also be applied in situations where the nonlinear element is a multiport element and where multiple frequencies are considered.

One important note on the application of this in-place averaging method is that the relevant averaged quantities correspond to the actual circuit waveforms. A particular nonlinear or time-varying circuit element may exhibit significantly different waveforms depending upon the circuit in which it is located. Hence, the replacement of the nonlinear element in the averaged circuit depends upon the remainder of the circuit. This phenomenon has been investigated in the case of low frequency averaging (i.e. index-0 averaging) in [3, 4].

Switch Equations Consider the switch element shown in Figure 3a. The relationship between its voltages and currents, as developed in [12], is given by

$$\begin{aligned} i_a &= u\,i_c \\ V_{cp} &= u\,V_{ap} \\ i_p &= (1-u)\,i_c \end{aligned} \tag{8}$$

where u takes on the values 0 and 1 corresponding to the two possible configurations. An equivalent representation using controlled sources is shown in Figure 3b. This three-terminal element can be imbedded in a two-port hybrid representation with controlling port variables taken as V_{ap} and i_c and non-controlling port variables i_a and V_{cp}. An averaged representation for this switch element can be obtained by first solving the circuit for the controlling variables V_{ap} and i_c in terms of independent source, capacitor voltage, and inductor current waveforms for each of the values of u. Then, one would need to evaluate $\langle u\,i_c\rangle_k$ and $\langle u\,V_{ap}\rangle_k$ for the index-k average.

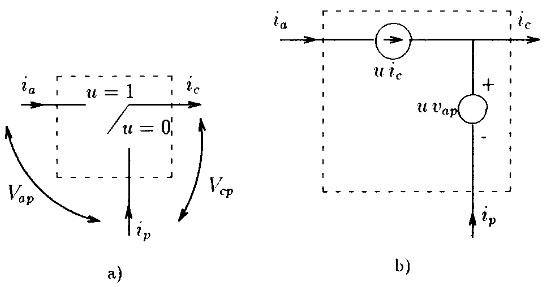

Figure 3: (a) Switch Element and (b) Equivalent Representation with Controlled Sources

The convolution formula

$$\langle xy\rangle_k = \sum_i \langle x\rangle_{k-i}\langle y\rangle_i \tag{9}$$

can be used to complete the calculation. This formula is applicable to the products of waveforms. For example, let us apply the averaging operation to the product $u\,i_c$ where we wish to consider only the index-0 and index-1 averages. Choosing these components is equivalent to approximating u and i_c by

$$u = \langle u\rangle_0 + \langle u\rangle_{-1}e^{-j\omega_s t} + \langle u\rangle_1 e^{j\omega_s t}$$
$$i_c = \langle i_c\rangle_0 + \langle i_c\rangle_{-1}e^{-j\omega_s t} + \langle i_c\rangle_1 e^{j\omega_s t}.$$

We obtain

$$\langle u\,i_c\rangle_0 = \langle u\rangle_{-1}\langle i_c\rangle_1 + \langle u\rangle_0\langle i_c\rangle_0 + \langle u\rangle_1\langle i_c\rangle_{-1}$$
$$\langle u\,i_c\rangle_1 = \langle u\rangle_0\langle i_c\rangle_1 + \langle u\rangle_1\langle i_c\rangle_0 .$$

In some cases nonlinear characteristics may be expressed as products of more than two signals. In this case we can group the term into a product of two lower order terms. The procedure can then be applied in a recursive manner until only products of linear terms remain.

Rectifier Bridge A similar approach can be taken for the full bridge rectifier shown in Figure 4a. In particular, it is possible to construct a hybrid representation for the equivalent two-port model shown in Figure 4b. Such a representation could be expressed as

$$\begin{aligned} i_s &= -sign(i_p)\,i_p = -abs(i_p) \\ v_p &= sign(i_p)\,v_s . \end{aligned} \tag{10}$$

The first step in constructing an averaged circuit model for a circuit with this element is to solve for the controlling port variables i_p and v_s in terms of independent source, capacitor voltage, and inductor current waveforms. The second step is to compute the relevant coefficients $\langle sign(i_p)\,i_p\rangle_k$ and $\langle sign(i_p)\,v_s\rangle_k$ where the explicit expressions for i_p and v_s from the first step are used.

The following section contains examples of PWM and resonant type circuits to illustrate the application of the method.

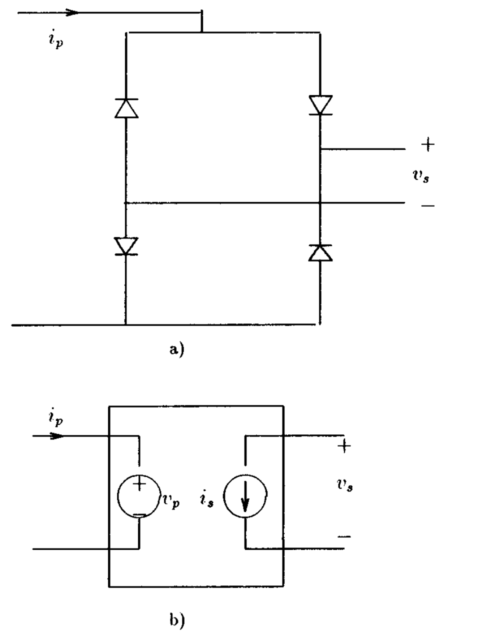

Figure 4: a) Rectifier Bridge and b) Equivalent Representation with Controlled Sources

3 Examples

PWM Up-Down Converter

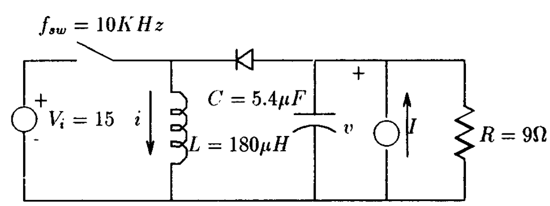

Figure 5: Up-Down Converter

To perform generalized in-place averaging on the up-down converter in Figure 5, we firstly note that for operation in continuous conduction mode, the diode and switch pair can be replaced by the three-terminal device shown in Figure 3a. For this example, the controlling port variables for the switch network can be taken as $V_{ap} = v - V_i$ and $i_c = i$, respectively. Secondly, we construct a sequence of *coupled* averaged subcircuits that correspond to the different average indices, as

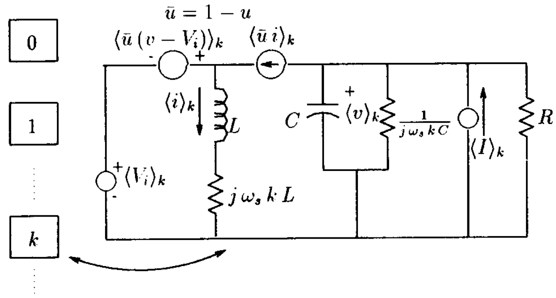

Figure 6: Generalized Averaged Model for an Up-Down Converter

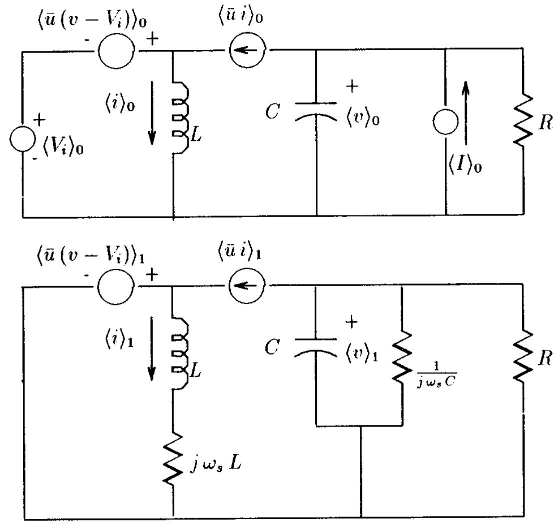

Figure 7: Averaged Up-Down Converter Model that Includes Index-0 and Index-1 Coefficients

shown in Figure 6. All components are replaced by their appropriate averaged models, as illustrated in this figure for the index-k subcircuit. In the case of the PWM switch pair, the convolution operation (9) is applied. Note that the coupling between the subcircuits is due to the nonlinear switch element.

The averaged circuit model can now be approximated by including only a small subset of the averaged subcircuits. In the simplest case we would include only the index-0 average. This would be equivalent to an approach based on state-space averaging. To obtain a greater degree of accuracy we could also include the index-1 average. In this case two subcircuits need to be included in the model, one for the index-0 average and one for the index-1 average. This model is shown in Figure 7. Note that the resulting averaged circuit is linear and time-invariant for open-loop operation since $\langle u \rangle_0$ and $\langle u \rangle_1$ are constant in this case.

As previously mentioned, one application of this generalized averaging procedure is in circuit based simulation. Since each index-k average waveform is complex valued for ($k \neq 0$), an ideal simulation package would be capable of handling complex valued circuit waveforms. If such a program is not available, the averaged circuit model must be separated into real and imaginary parts. The relatively simple nature of this process lends itself to automation, and an *awk* script was written to perform these manipulations on general SPICE3 circuit decks. It converts the original circuit input deck to an averaged deck including arbitrary user-requested index-k averages (as well as automatically creating the real and imaginary subcircuits). Averages of non-linear components can be imbedded in the original spice deck as comments, permitting the original deck to be used as 'source' for the averaged circuit representation.

To illustrate the application, we generated a set of numerical simulations using Berkeley Spice version 3D2. The averaged model of Figure 7 (with $I = 0$) has been simulated for a start-up transient with a duty cycle of 75%. The reconstructed inductor current (labeled i01_01) is shown in Figure 13. The figure also shows a waveform that corresponds to the index-0 component of the averaged inductor current (labeled i0_01) that results when both index-0 and index-1 coefficients are included. For purposes of comparison, this figure also shows analogous inductor current waveforms that correspond to the underlying switched circuit (labeled i_{actual}) and to a simpler model that considers only the index-0 coefficient (labeled i0_0).

Since the duty cycle differs from 50%, second harmonics are nonzero in the PWM circuit. In this case greater accuracy is obtained by including index-0,1, and 2 averages in our model. Figure13 also shows the results obtained for an averaged PWM circuit for which the index-0,1 and 2 coefficients were included. The trace labeled i012_012 is the reconstructed current from the index-0,1, and 2 coefficients. The current corresponding to the index-0 coefficient (i0_012) obtained with the same model is also shown. As is evident, including more coefficients in the model offers improved accuracy over simpler models which include fewer.

Each additional subcircuit in the averaged model introduces greater complexity, and increased simulation time. Table 1 compares the simulation times obtained for various circuits. As can be seen from the table, an averaged circuit model typically leads to a faster simulation than that obtained with the underlying circuit. All simulations were performed on a Dec3100 running Berkeley Spice 3D2 under ultrix.

Note that finding the steady-state DC operating point of

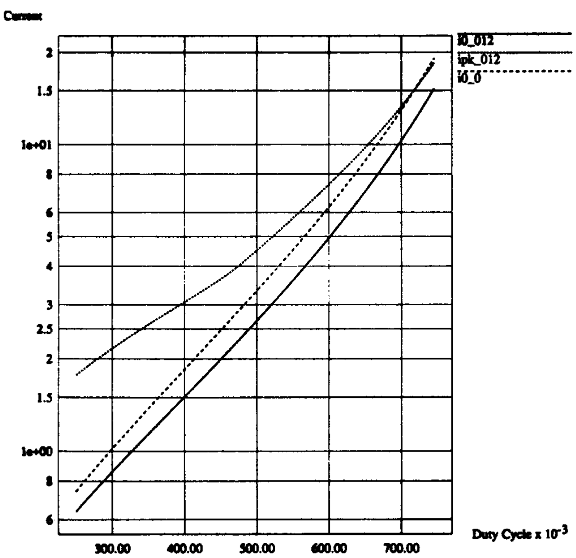

Figure 8: Steady-State solution and Estimate of RMS ripple for various averaged models

the PWM converter in the averaged domain is equivalent to finding the approximate steady state of the actual PWM circuit. A simple DC sweep of driving voltages in the averaged domain can then be used to find the steady state values of inductor current as well as an estimate of ripple for different duty cycles. This result is shown graphically in Figure 8. This figure shows the DC values of the inductor current (i0_012) as well as an estimate for the RMS ripple (ipk_012) for the averaged model that includes the index-0, -1, and -2 coefficients. For comparison purposes this graph also shows the inductor current resulting from a model indluding only the index-0 average i0_0. Note that this last result corresponds exactly to what could be obtained using state-space averaging methods.

Resonant Converter

An averaged circuit model for the series resonant converter of Figure 9 is shown in Figure 10. This averaged model is

Circuit	CPU Time
Actual Up-Down Circuit	.53s
Index-0 averaged circuit	.15s
Index-0, index-1 averaged circuit	.21s
Index-0, index-1, and index-2 averaged circuit	.45s

Table 1: Simulation Times for Various Circuits

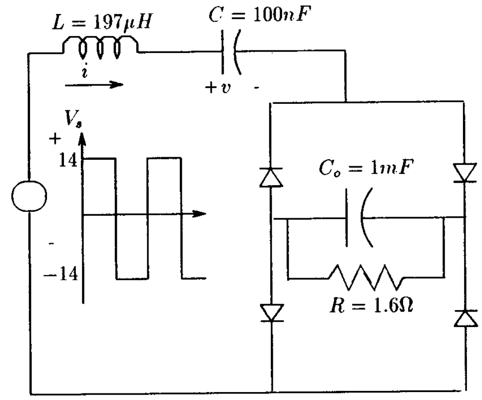

Figure 9: DC-DC Series Resonant Converter

built out of a set of *coupled* subcircuits that corresond to the different average indices. The coupling is due to the rectifier bridge. The model can be approximated by omitting subcircuits and portions of subcircuits that do not participate significantly in the operation. In particular, Figure 11 shows an approximate averaged model that retains only the AC side of the index-1 subcircuit and the DC side of the index-0 subcircuit. This approximation corresponds precisely to that made in [10] for this example.

Once the averaged circuit is created, we can readily obtain a closed-form solution for the steady-state operating point by using DC analysis (replacing capacitors with open circuits, and inductors with shorts).

Simulation waveforms for the approximate averaged model are compared to waveforms for the underlying circuit in Figure 12. The graphs plot on the same axes the waveforms for the underlying circuit and those for the averaged circuit, for

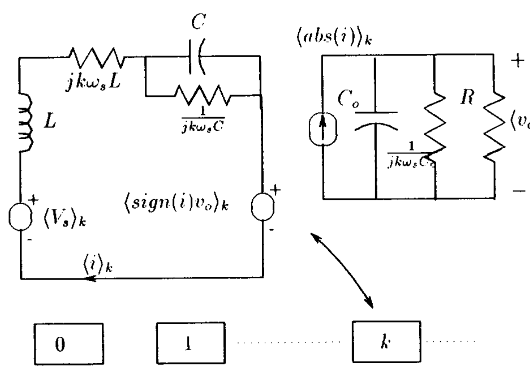

Figure 10: Averaged Circuit Model for DC-DC Series Resonant Converter

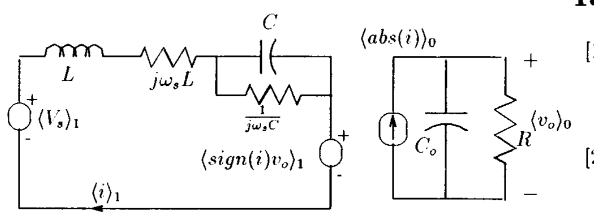

Figure 11: Approximate Averaged Model for DC-DC Series Resonant Converter

a step change in drive frequency from 40Khz to 38Khz.

4 Conclusions

A method of in-place averaging which is applicable to resonant circuits as well as PWM circuits has been introduced. The technique yields circuit models suitable for entry into simulation programs such as SPICE, and is easily automatable. Arbitrary degrees of accuracy can be obtained by including higher order averages in the procedure. Approximate solutions for steady-state operation of many circuits can easily be obtained by simply applying KVL and KCL to the averaged circuit. Ripple estimates can also be easily obtained.

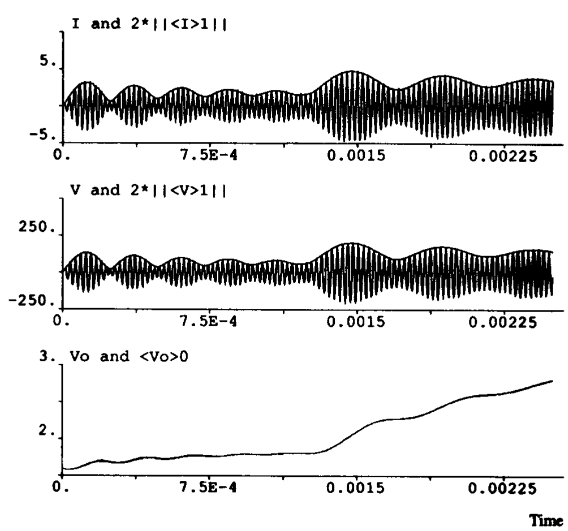

Figure 12: Comparison of Simulation Waveforms for the Approximate Averaged Model and for the Underlying DC-DC Resonant Converter

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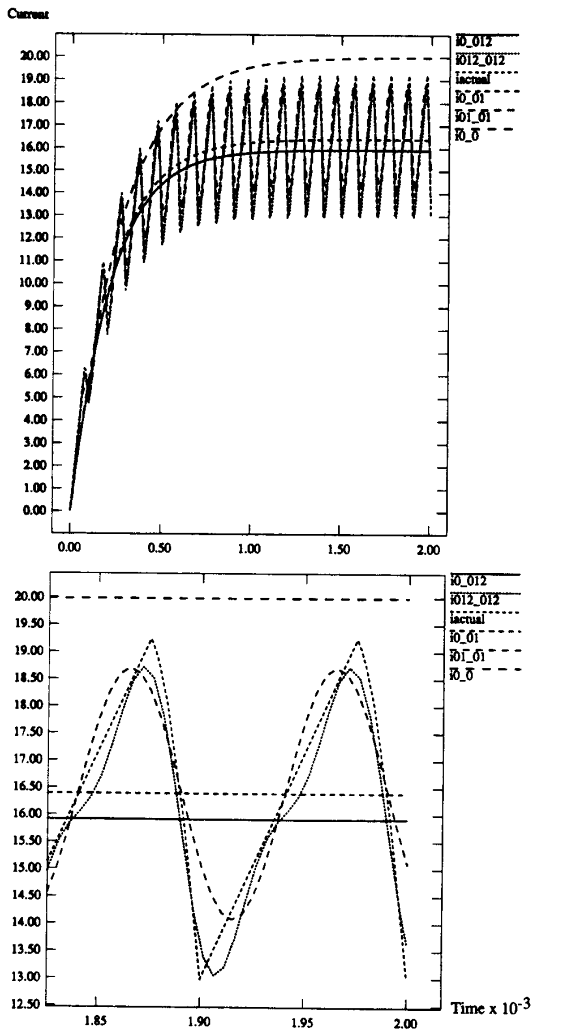

Figure 13: Comparison of Simulation Results Obtained with the Generalized Averaged Model of Up-Down Converter including 0,1,2 averages (i012_012 and i0_012), the 0,1 averages (i01_01 , i0_01), the Underlying Switched Model (i_{actual}), and the Averaged Model which Considers Only Index-0 Coefficients (i0_0)